Maths Module 1

An Introduction to Mathematics

This module covers concepts such as:

- basic arithmetic
- rounding
- order of operations
- mental computation strategies

www.jcu.edu.au/students/learning-centre

TheLearningCentre
UNLOCK YOUR POTENTIAL

Module 1

An Introduction to Mathematics

1. Introduction to Arithmetic
2. Arithmetic of Whole Numbers
3. Working with Decimals
4. Rounding and estimating
5. Order of Operations
6. Answers
7. Helpful Websites
8. Mental Computation Strategies to Explore

1. Introduction to Arithmetic

- Mathematics is the science of patterns and relationships related to quantity. For example, below are some relationships of the many you may have come across:
  - Every number is related to every other number in a number relationship. For example, 8 is 2 less than 10; made up of 4 and 4 (or 3 and 5); and is 10 times 0.8; is the square root of 64; and so on….
  - Number relationships are the foundation of strategies that help us remember number facts. For instance, knowing 4 + 4 = 8 allows one to quickly work out 4 + 5 = 9 (one more than 8); If one knows that 2 x 5 = 10, then 4 x 5 and 8 x 5 can easily be calculated (double 2 is 4 and so double 10 is 20; then double 4 is 8 and so double 20 is 40).
  - Each digit in a written numeral has a 'place' value which shows its relationship to '1'. For example, in 23.05 the value of the '2' is 20 ones, while the value of the '5' is only five-hundredths of one. Understanding place value is critical when working with numbers.

- Mathematics is considered a universal language; however, words in English can often have more than one meaning which is why we sometimes find it difficult to translate from English to mathematical expressions.

- Arithmetic is a study of numbers and their manipulation.

- The most commonly used numbers in arithmetic are integers, which are positive and negative **whole** numbers including zero. For example: -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6. Decimal fractions are not integers because they are 'parts of a whole', for instance, 0.6 is 6 tenths of a whole.

- The symbols we use between the numbers to indicate a task or relationships are the **operators**, and the table below provides a list of common operators. You may recall the phrase, 'doing an operation.'

Symbol	Meaning
$+$	Add, Plus, Addition, Sum
$-$	Minus, Take away, Subtract, Difference
$\times$	Times, Multiply, Product,
$\div$	Divide, Quotient
$\pm$	Plus and Minus
$\|a\|$	Absolute Value (ignore negative sign)
$=$	Equal
$\neq$	Not Equal
$<$	Less than
$>$	Greater than
$\ll$	Much Less than
$\gg$	Much More than
$\approx$	Approximately equal to
$\leq$	Less than or equal
$\geq$	Greater than or equal

Example Problems:

1. $12 \pm 6 = 6 \; and \; 18. \quad 12 + 6 \; and \; 12 - 6$
2. $|-6| = 6$
3. $12.999 \; \approx 13$
4. $12 \; \neq 7$
5. Drink driving is a blood alcohol level of ≥ 0.05
6. Speed of light $\gg$ speed of sound

1. Your Turn:

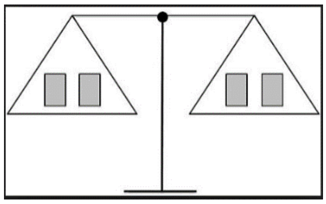
Are the following statements true?

a. $32 \neq \; 4 \times 8$
b. $7 > 6$
c. $4 \geq 4$
d. $|5| = 5$
e. $37.1 + 22.02 = 59.3$

2. Arithmetic of Whole Numbers

Integers

Whole numbers are integers; there are positive and negative integers. Positive integers are 1, 2, 3, 4, 5… The negative integers are … -5, -4, -3, -2, -1 (the dots before or after the sequence indicate that there are more numbers in this sequence that continue indefinitely).



Here are some more terms for you:

An **equation** implies that what is on either side of the **'='** sign balances**.**
The **sum** of two numbers implies two numbers are added together.
The sum of 4 and 8 is 12; $4 + 8 = 12$
The **difference** of two numbers implies that the second number is subtracted from the first number.
The difference between 9 and three is 6; $9 - 3 = 6$
The **product** of two numbers implies that two numbers are multiplied together.
The product of 3 and 4 is 12; $3 \times 4 = 12$
The **quotient** of two numbers implies that the first number is divided by the second.
The quotient of 20 and 4 is 5; $\frac{20}{4}$ = 5 or $20 \div 4 = 5$

Rational Number: The term rational derives from the word ratio. Hence, a rational number can be a described by a ratio of integers or as a fraction. For example, $\frac{3}{4} \; and \; 0.75$ are both rational numbers.

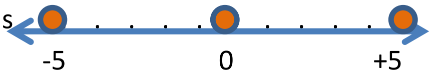
Irrational number: A number that cannot be written as a simple fraction or as a decimal fraction. If the number goes on forever without **terminating**, and without **repeating**, then it is an irrational number. For example, π is a recurring decimal that does not repeat: 3.14159… Therefore, π is an irrational number.

Factors: A whole number that divides exactly into another number.
$1, 2, 3, 5, 6, 10, 15, 30 \; are \; factors \; of \; 30$

Multiple: A number that can be divided by another number without a remainder.
$3, 6, 9, 12, 15 \; are \; all \; mulitples \; of \; 3$

Directed Numbers (negative and positive integers)

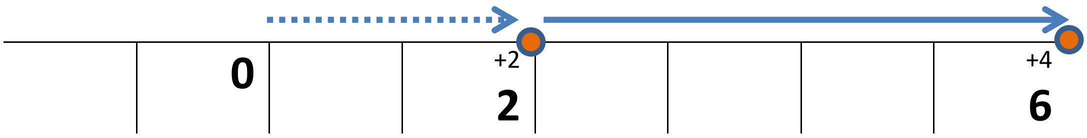
Directed numbers are like arrows with a particular and direction $+$ and $-$ They have positive and negative signs to signify their direction. Note that when using the calculator we use the $(-)$ key rather than the subtraction key. Each negative number may need to be bracketted $(-3) \; + \; (+3) \; = \; 0.$
When naming directed numbers, we use the terms *negative* and *positive* numbers; avoiding the terms *plus* and *minus* unless you are indicating that an operation is taking place of addition and subtraction.
Thus $(-3) \; + \; (+3) \; = \; 0$ reads '*negative 3 plus positive 3 equals zero'.*

To use a graphic symbol we can display (-5) and $(+5)$ as 

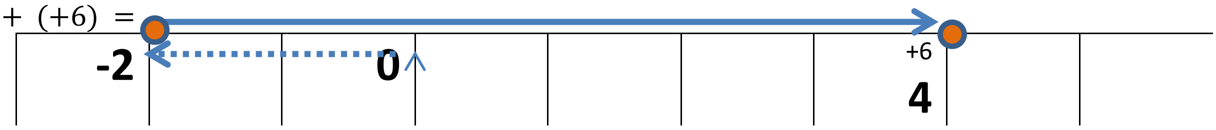
This graphic symbol is known as a number line and can be used to show how and why operations work.

Addition: To add a number we move to the right:

$2 + 4 = 6$

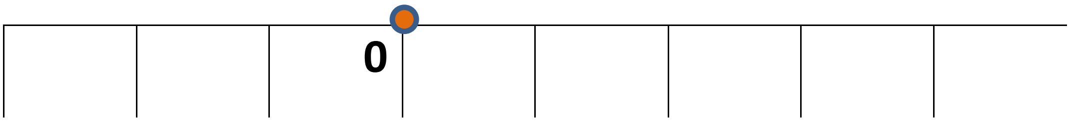


$(-2)\ +\ (+6)\ =$



In this example we added a positive number beginning at a negative number. So how it works – start a zero, move in a negative direction two places, then move in a positive direction six places; we end at four.

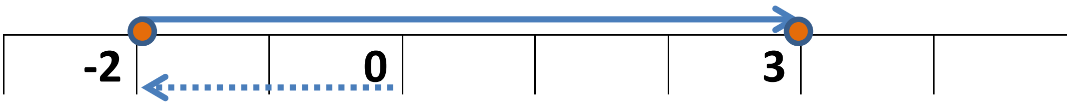
Your turn to represent $0.5\ +\ 3\ =$



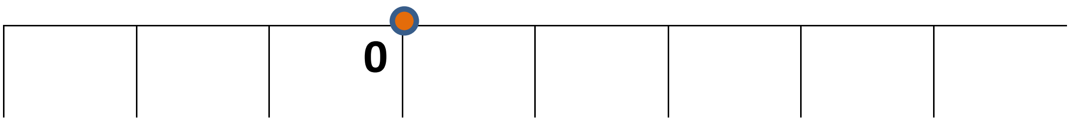
Subtraction:
To subtract a positive number we move that number of places to the left. For example, $3 - 4 = (-1)$; we start at zero, move three spaces to the right (in a positive direction), then move four spaces to the left (in a negative direction) to end up at negative one. We can think about this as if we were on a lift. If we start at the ground floor and go up three floors, then down four floors, we would be one level below ground.

Remember that to subtract an integer means adding its opposite. Hence, to subtract a negative number we move to the right rather than the left – hence in a positive direction.

For example: $(-2) - (-5) = 3$; 'negative 2 minus negative 5' which means $(-2) + 5$ (adding the opposite)



Your turn to represent $(+2) - (+5)\ =\ (-3)$



Two key points:

- Subtracting a negative number is the same as adding its opposite. $4 - (-3) = 4 + 3 = 7$
- Adding a negative number is the same as subtracting a positive number. $4 + (-2) = 4 - 2 = 2$

2. Your Turn

a. Find the sum of 3, 6 & 4

b. Find the difference of 6 and 4

c. Find the product of 7 & 3

d. Find the quotient of 20 and 4

e. Find the factors of 24

f. Find some multiples of 7

Refreshing your memory of traditional methods for arithmetic

Addition

$$\begin{array}{r} {}^{1} \\ 47 \\ +\,86 \\ \hline 133 \end{array}$$

- The first step is to add the ones and we get 13ones or 3ones and 1ten, so we add the 1ten to the tens then we add 5tens and 8 tens to get 13tens or 130+3

Subtraction

- 74 – 36 = we can read 74 as 7*tens* and 4*ones* or 6*tens* and *14ones*. This is called **decomposing** numbers.

$$\begin{array}{r} {}^{6\;\;14} \\ \cancel{74} \\ -36 \\ \hline 38 \end{array}$$

so first we cannot take 6 from 4, so we decompose
now we can subtract 6 *ones* from 14*ones*
then we take 3 *tens* from 6 *tens*

Division

Traditional bring-down method

```
70          1
           5 8 = 59 R 25
     ________
  63) 3 7 4 2
      3 1 5
      -----
        5 9 2
        5 0 4
        -----
          8 8
          6 3
          ---
          2 5
```

This is a method of long division you may recall. It was taken from the following book which covers various computation strategies. You will notice the 70 in a bubble; this reflects an easy number to work with to estimate 374 ÷ 63.

Van de Walle, J. A. (2007). *Elementary and middle school mathematics: Teaching developmentally* (6th ed.). Sydney: Pearson Education.

Multiplication

- Let's look at 47 × 65
- Let's estimate first 50 × 60 = 3000

$$\begin{array}{r} {}^{4\,3} \\ 47 \\ \times\,65 \\ \hline 235 \\ 2820 \\ \hline 3055 \end{array}$$

multiply the **ones**, 7×5=35 rename 3tens and 5ones
5 × 4 is 20tens, add 3tens = 23 tens,
rename: 2hundreds 3tens and five ones
multiply the **tens**, 6tens × 7ones = 42tens, 4 hundreds & 2tens
6 tens × 4 tens are 24 hundreds plus 4 hundreds = 28
then add 235+2820=3055

2. Your Turn:

Calculate the following:

g. $153 + 69$

h. $132 - 78$

i. $953 \div 18$

j. 692×36

Sometimes when writing a product calculation, the multiplication sign 'x' is replaced with a dot '.' or '*' or it can be omitted.

For example:
$3 \times 6 \times 9$ could be written as 3.6.9 or as $(3)(6)(9)$
$3(4 + 5)$ can also be written as $3 \times (4 + 5)$

This table will assist you multiplication and division of positive and negative numbers.

Rules for multiplying and dividing positive and negative integers	
(positive) x (negative) = negative	(positive) ÷ (negative) = negative
(negative) x (positive) = negative	(negative)÷ (positive) = negative
(negative) x (negative) = positive	(negative)÷ (negative) = positive
(positive) x (positive) = positive	(positive)÷ (positive) = positive

2. Your Turn:

k. $3 \times (-2) =$

l. $(-1) \times 7 =$

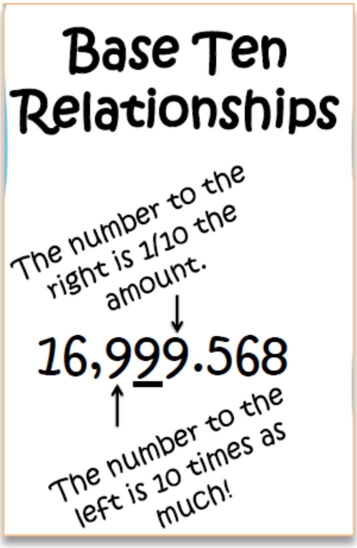
m. $(-2)(-4) =$

n. $12 \div (-4) =$

o. $\frac{-8}{4} =$ (tip: another way to signify division)

3. Working with Decimals

Key Ideas:



The decimal separates whole numbers from parts of a whole.

For instance, 3.6; three is the whole number and 6 tenths of a whole.

Each digit in a number has a 'place value' (related to one).
The value depends on the position of the digit in that number.
Each position can be thought of as columns.
Each column is a power of ten.
For example, let's look at 56.39

Tens	Ones	Tenths	Hundredths
1x10	1	$\frac{1}{10}$	$\frac{1}{100}$
5	6	3	9

$5 \times 10^1 = 50$

$6 \times 1 = 6$

$3 \times \frac{1}{10} (3 \times 10^{-1}) = 0.3$

$9 \times \frac{1}{100} (9 \times 10^{-2}) = 0.09$

A **recurring decimal** is a decimal fraction where a digit repeats itself indefinitely
For example, two thirds = 0.666666
Because the number repeats itself from the tenths position a dot can be written above the 6 as such $0.\dot{6}$
If the number was one sixth($\frac{1}{6}$), which as a decimal is 0.16666, then we signify as $0.1\dot{6}$
If the number contained a cluster of repeating digits, for example, five elevenths = 0.454545 we write $0.\dot{4}\dot{5}$
A terminating decimal is a number that terminates after a **finite** (not infinite) number of places, for example:

$\frac{2}{5} = \frac{4}{10}$ or 0.4; and $\frac{3}{16} =$ 0.1875 (terminating after 5)

Working with Decimals

What happens when we multiply or divide by ten, or powers of ten?

Patterns are identified when multiplying by ten. However, often it is said that we simply add a zero. Is this correct? Think about 4.3×10, does it equal 4.30?

Another misconception is that when multiplying by ten we move the decimal one place, whereas, it is actually the digits that move.

When we multiply by ten, all digits in the number become ten times larger and they move to the left.
What happens when we divide by 10? All the digits move to the right.
$963.32 \div 10 = 96.332$ all of the digits more one place to the right.
$963.32 \times 10 = 9633.2$ all of the digits move one place to the left.

Another key point when working with addition or subtraction of numbers is to line up the decimal points. A zero can be regarded as a holder. For example, $65.32 + 74.634 =$ (see right)

$$\begin{array}{r} 65.320 \\ +\quad 74.634 \\ \hline 139.954 \end{array}$$

4. Rounding and Estimating

Rounding numbers is a method of decreasing the accuracy of a number to make calculations easier. Rounding is important when answers need to be given to a particular degree of accuracy. With the advent of calculators, we also need to be able to estimate a calculation to detect when an answer may be incorrect.

For example: *What is 7 divided by 9 to 3 decimal places?*

Some Rules for Rounding:

1. Choose the last digit to keep.
2. If the digit to the right of the chosen digit is 5 or greater, increase the chosen digit by 1.
3. If the digit to the right of the chosen digit is less than 5, the chosen digit stays the same.
4. All digits to the right are now removed.

So we know that 7÷9 =0.7777777777777777777777777777777777777etc.
The chosen digit is the third seven.
The digit to the right is 7 which is larger than 5, so we change the digit to 8.
$\therefore 7 \div 9 = 0.778$ to three decimal places.

Estimating is a very important ability which is often ignored. A leading cause of getting math problems wrong is because of entering the numbers into the calculator incorrectly. Hence, if you can estimate the answer first, you will be able to check if your calculations are correct. If not, something may have been entered incorrectly.

Some simple methods:

- **Rounding:** $273.34 + 314.37 = ?$ If we round to the tens we get $270 + 310$ which is much easier and quicker. We now know that $273.34 + 314.37$ should equal approximately 580.
- **Compatible Numbers:** $529 \times 11 = ?$ If decrease 11 to 10, then we can easily solve $529 \times 10 = 5290$. To calculate further we simply need to add 529 (one more group of 529).
- **Cluster Estimation:** $357 + 342 + 370 + 327 =$ All four numbers are clustered around 350, some larger, some smaller. So we can estimate using $350 \times 4 = 1400$

EXAMPLE PROBLEMS:

1. Round the following to 2 decimal places:
   a. 22.6783 gives 22.68
   b. 34.6332 gives 34.63
   c. 29.9999 gives 30.00
2. Estimate the following:
   a. $22.5684 + 57.355,\ so\ 23 + 57 \approx 80$
   b. $357 \div 19,\ so\ 360 \div 20 = 18$
   c. $27 + 36 + 22 + 31,\ so\ 30 \times 4 \approx 120$

4. Your Turn:

a. Round the following to 3 decimal places:
   i. $34.5994 \approx$
   ii. $56.6734 \approx$

b. Estimate the following:
   i. $34 \times 62 \approx$
   ii. $35.9987 - 12.76 \approx$
   iii. $35 + 32 + 27 + 25 \approx$

5. Order of Operations

Look at the example problem below

$$3 + 6 \times 2 = ?$$

If I do the addition, then the multiplication, the answer would be: $9 \times 2 = 18$
If I do the multiplication, then the addition, the answer would be: $3 + 12 = 15$
There can NOT be two answers to the same question. A rule is required to make sure everyone uses the same order.
There is a **calculation priority sequence** to follow. Different countries, different states, even different teachers use different mnemonics to help you remember the order of operations, but common versions are BOMDAS or BIMDAS which stand for:

Brackets	{[( )]}
Other or **I**ndices	$x^2, \sin x, \ln x, etc$
Multiplication or **D**ivision	$\times$ or $\div$
Addition or **S**ubtraction	$+$ or $-$

The Calculation Priority Sequence:

1. Follow the order (BOMDAS or BIMDAS)
2. If two operations are of the same level e.g. $(\times\ or\ \div)\ or\ (+\ or\ -)$, you work from left to right.
3. If there are multiple brackets, then work from the inside set of brackets outwards.

Example Problems:

1. Solve: $5 + 7 \times 2 + 5^2$

Step 1: 5^2 has the highest priority so: $5 + 7 \times 2 + 25$

Step 2: 7 x 2 has the next priority so: $5 + 14 + 25$

Step 3: only addition left so left to right: $19 + 25$

44

$\therefore 5 + 7 \times 2 + 5^2 = 44$

2. Solve: $[(3 + 7) \times 6 - 3] \times 7$

Step 1: $(3 + 7)$ has the highest priority: $[10 \times 6 - 3] \times 7$

Step 2: 10×6 inside brackets has the next priority: $[60 - 3] \times 7$

Step 3: the brackets and then the multiplication: 57×7

$= 399$

$\therefore [(3 + 7) \times 6 - 3] \times 7 = 399$

5. Your Turn:

a. $4 \times (5+2)+6-12 \div 4=$

b. $3 \times 7+6-2+4 \div 2+7=$

c. $10+4 \times 6=$

d. $(10+4) \times 6=$

e. $50-6 \times 8=$

f. $(50-6) \times 8=$

g. $200-150 \div 20=$

h. $(200-150) \div 20=$

i. $2.4-0.8 \times 5+8 \div 2 \times 6=$

j. $(2.4-0.8) \times 5+8 \times 6 \div 2=$

k. $60+40-30 \times 2+8 \times 5=$

l. $60+(40-30) \times (2+8) \times 5=$

m. $\frac{120}{3 \times 4}=$

Tip: When working the examples given, use a lot of space to set your working out as a series of logical steps. This will assist your mathematical thinking and reasoning.

6. Answers

1. a. False b. True c. True d. True e. False

2. a. 13 b. 2 c. 21 d. 5 e. 1, 2, 3, 4, 6, 8, 12, 24
   f. 14, 21, 28… g. 222 h. 54 i. $52.9\dot{4}$ or 52r17 or $52\frac{17}{18}$
   j. 24912 k. -6 l. -7 m. 8
   n. -3 o. -2

3. No questions for section three

4. a. (i) 34.599 a. (ii) 56.673
   b. (i) $\approx$ 35x60=2100 b. (ii) $\approx$ 36-13=23 b. (iii) $\approx$ 30x4=120

5. a. 31 b. 34 c. 34 d. 84 e. 2
   f. 352 g. 192.5 h. 2.5 i. 22.4 j. 32
   k. 80 l. 560 m. 10

7. Helpful Websites

Integers: http://www.factmonster.com/ipka/A0876848.html

Rounding: http://www.mathsisfun.com/rounding-numbers.html

Estimating: http://mathandreadinghelp.org/how_to_estimate_a_math_problem.html

Order of Operations: http://www.mathgoodies.com/lessons/vol7/order_operations.html

8. Mental Computation Strategies

On the following pages are some mental computation strategies to explore. Developing strategies to work and solve basic number facts mentally is critical for future mathematical thinking and reasoning. You may have some others that you use and may be willing to share. If so, please email any suggestions to kerry.smith@jcu.edu.au

The suggested strategies below have been adapted from:
Anderson, J., Briner, A., Irons, C., Shield, M., Sparrow, L., & Steinle, V. (2007), *The Origo handbook of mathematics education* (Section 3.1). Australia: Origo Education.

Addition

Adding the places

See

- 35 + 23 = ____

Think

- 35 + 20 + 3

- 7 + 5 = 12
addend addend sum

Bridging to ten

See

- 17 + 8 = ___

Think

- 17+3+5

Compensating

See

- 28+36___

Think

- (30+36)-2

Counting on

See

- 58+24=___

Think

- 58+10+10+4

Using compatible addends

See

- 14+23+16=___

Think

- 14+16+23

(using compatible pairs; rainbow facts)

Using doubles

See

- 7+8=

Think

- 7+8=15

Because 7+7=14

Addition

Using place value

See
- 56+17=___

Think
- 50+10+6+7

Or 6+7+50+10

- Expand the addends to places, or this can be called a 'split strategy' – breaking the numbers into more manageable bits

Subtraction

Adjusting

See 83-58=___

Think:
80-55 (subtract 3 from each)

Bridging to ten

See 45-12=___

Think:
45-5-7

compensating

See 82-35=___

Think:
(80-35)+2

Counting back

See 86-24=___

Think:
86-10-10-4

Subtraction

Subtracting the places

See 68-35=____

Think:

68-30-5

Thinking addition

See 15-12

Think:

12+3 is 15, so

15-12 is 3

Using place value

12 − 7 = 5

Minuend subtrahend difference

See 56-32=___

Think:

(50-30)+(6-2)

Expanding the minuend and subtrahend otherwise thought as the split strategy

multiplication

3 x 4 = 12

Multiplicand multiplier product

Factors

Breaking a factor

See 26x4=____

Think:

(25x4)+(1x4)

Splitting into manageable bits (not places)

multiplication

Building down	Building up
See 19x3=___ Think: 20x3=60, so 19x3=57 (60-3)	See 21x3=___ Think: 20x3=60, so 21x3=63 (60+3)
factorising	**Recognising midpoints**
See 15x7 Think: 3x5x7 or 5x3x7	See 8x15 Think: 8x15 must be halfway between 8x10 (80) and 8x20 (160)
Doubling	**Doubling and halving**
See 8x14 Think: double 14 is 28 double 28 is 56 double 56 is 112	See 12x15=___ Think: 6x30 (use an array, or blocks, to demonstrate this strategy)

multiplication

Using a benchmark number

See 32x5=___

Think:

32x10=320, so 32x5 must be one half of 320 (160)

See 16x25=___

Think:

16x100=1600, so 16x25 must be one quarter of 1600 (400)

Using division

See 1/5 x 30=___

Think:

30÷5

Using place value

- See 26x3=___

Think:

(20x3) + (6x3)

Division

12 ÷ 3 = 4

Dividend divisor quotient

adjusting

See 75 ÷5=___

Think:

150 ÷10 (double each number)

Division

Breaking the dividend

See 138 ÷3=___

Think:
(120 ÷3) + (18 ÷3)

halving

See 112 ÷8=____

Think:
half of 112 is 56
half of 56 is 28
half of 28 is 14

Thinking multiplication

See 35 ÷7=___

Think:
7x5=35, so 35 ÷7is 5

Using place value

See 318 ÷3=___

Think:
(300 ÷3) + (18 ÷3)

Reference

Anderson, Briner, Irons, Shield, Sparrow, Steinle (2007), *The ORIGO handbook of mathematics education*. Australia: PrintPoint