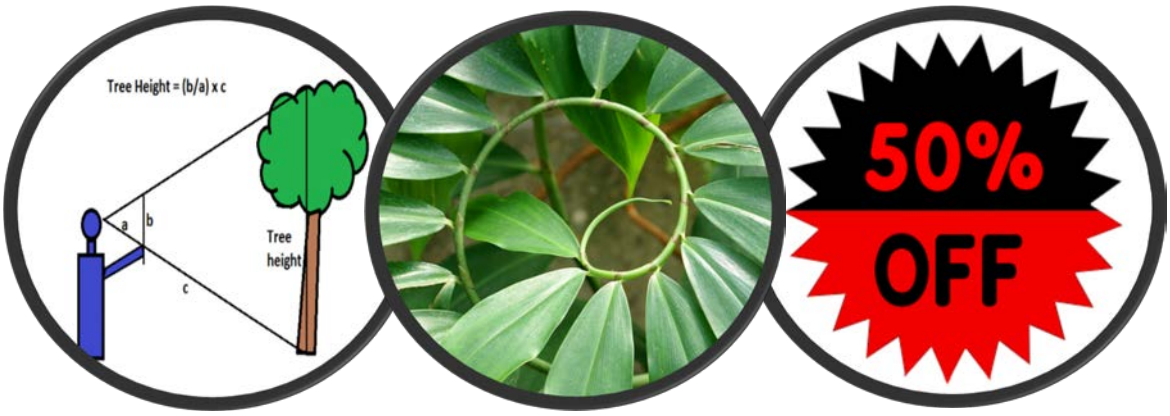
Maths Module 4

Powers, Roots and Logarithms

This module covers concepts such as:

- powers and index laws
- scientific notation
- roots
- logarithms



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Module 4

Powers, Roots, and Logarithms

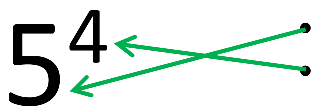
1. Introduction to Powers

Powers are a method of simplifying expressions.

- An equation such as : $7+7+7+7+7+7+7+7+7=63$ could be simplified as: $7\ \times\ 9 =\ 63$
- Whereas an expression such as: $7\times7\times7\times7\times7\times7\times7\times7\times7$ could be simplified as: 7^9

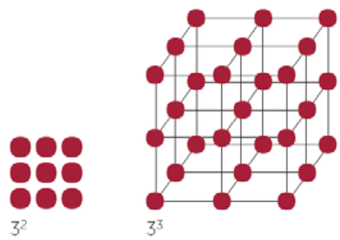
A simple way to describe powers is to think of them as how many times the base number is multiplied by itself.

- $5\times5\times5\times5$ is the same as 5^4 is the same as 625



The 5 is called the base.
The 4 is called the exponent, the index or the power.

- The most common way to describe this expression is, ‘five to the power of four.’
- The two most common powers (2 & 3) are given names: 3^2 is referred to as ‘3 squared’ and 3^3 as ‘3 cubed.’
- However, note that -3^2 is different to $(-3)^2$; the first is equivalent to (-9), whereas the second is equivalent to $(+9)$ In the first example, -3^2, only the 3 is raised to the power of two, in the second, $(-3)$is rasied to the power of two, so $(-3)\times(-3)=9$, ‘positive’ number because $(-)\times(-)=(+)$



- Whereas, a negative power, such as 6^{-3}, reads as ‘six raised to the power of negative three’, it is the same as $\frac{1}{6^3}$: it is the reciprocal. $\therefore\ 6^{-3}=\frac{1}{6^3}=\frac{1}{216}$
- To raise to the negative power means one divided by the base raised to the power, For example, 6^{-3} can be read as ‘one divided by the 6 raised to the power of three’ (six cubed).
- Another example: $\frac{1}{4^3}$ is the same as 4^{-3} , which is also $\frac{1}{64}$
- Take care when working with powers of negative expressions: $(-2)^3 =$
  $(-8)\ negative\ times\ negative\ times\ negative = negative\ (-\times-\times-=-)$
  $(-2)^4=16\ (-\times-)(-\times-)=+$
  $\therefore\ (-x)^5$ will be a negative expression, whereas $(-x)^6$ will be positive.

It helps if you can recognise some powers, especially later when working with logarithms

* The powers of 2 are: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 2048, 4096, 8192,…
* The powers of 3 are: 3, 9, 27, 81, 243, 729,…
* The powers of 4 are every second power of 2
* The powers of 5 are: 5, 25, 125, 625, 3125,…
* The powers of 6 are: 6, 36, 216, 1296,…
* The powers of 7 are: 7, 49, 343, 2401,…
* The powers of 8 are: every third power of 2
* The powers of 9 are every second power of 3
* The powers of 10 are: 10, 100, 1000, 10 000, 100 000, 1 000 000,…
* The powers of 16 are every fourth power of 2

1. Your Turn:

Work out the value of the following based on what you have understood from the introduction:

a. $2^3 =$

b. $9^2 =$

c. $\left(\frac{1}{2}\right)^3 =$

d. $(-4)^2 =$

e. $(-3)^3$

f. $(-\frac{1}{4})^2 =$

g. $0^3 =$

h. $(-0.1)^2 =$

i. $(0.5)^3 =$

j. $4^{-2} =$

k. $1^{-11} =$

l. $4^{-1} =$

m. $(-4)^{-1} =$

n. $(0.5)^{-4}$=

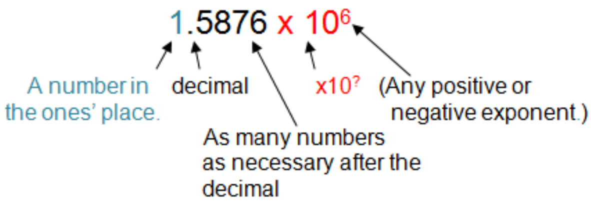
o. $(\frac{3}{4})^{-3} =$

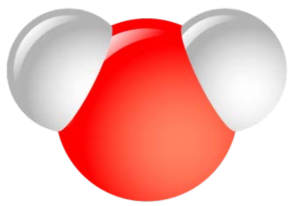
p. $(-2)^{-3} =$

2. Scientific notation

- The most common base is 10. It allows very large or small numbers to be abbreviated. This is commonly referred to as 'scientific notation'.
    - o $10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$. (Note: $100\,000$ has 5 places (as in place value) behind '1', a way to recognise 10 to the power of five
    - o $10^{-4} = \frac{1}{10^4} = 10 \div 10 \div 10 \div 10 = 0.0001$. (Note: 0.0001 has 4 places (as in place value) in front; one divided by 10 to the power of four)
    - o $3000 = 3 \times 1000 = 3 \times 10^3$
    - o $704\,500\,000 = 7.045 \times 100\,000\,000 = 7.045 \times 10^8$

- A classic example of the use of scientific notation is evident in the field of chemistry. The number of molecules in 18 grams of water is 602 000 000 000 000 000 000 000 which is written as 6.02×10^{23} which is much easier to read and verbalise.





2. Your Turn:

Write the following in scientific notation:

a. 450

b. 90000000

c. 3.5

d. 0.0975

Write the following numbers out in full:

e. 3.75×10^2

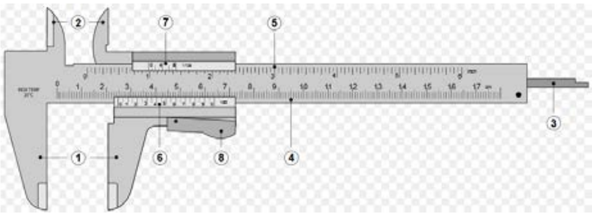
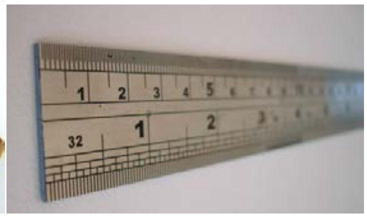
f. 3.97×10^1

g. 1.875×10^{-1}

h. -8.75×10^{-3}

3. Significant Figures

Scientific notation is used for scientific measurements as is **significant figures**. We briefly touched on this concept in module one. Both scientific notation and significant figures are used to indicate the accuracy of measurement. For example, if we measured the length of an item with an old wooden ruler, a steel ruler, and then some steel callipers, the measurements would vary slightly in the degree of accuracy.

Perhaps the item might be approximately 8cm long. Hence:

- The wooden ruler will measure to the nearest cm $\pm 0.5cm$, 8.2cm; 2 Significant Figures (sig. figs.)
- The steel ruler will measure to the nearest 0.1 cm $\pm 0.05cm$, perhaps 8.18cm; 3 sig. figs.
- The callipers may measure to the nearest 0.01cm $\pm 0.005cm$, for instance 8.185cm; 4 sig.figs.

Some rules apply:

1. Any non-zero digit is significant. The position of the decimal point does not matter. For example:

14.8	3 sig. figs
148	3 sig. figs
1.48	3 sig. figs
0.148	3 sig. figs
1.4869	5 sig. figs

2. If there are zeros between numbers, they are significant. For example:

1.08	3 sig. figs
10.48	4 sig. figs
1.0048	5 sig. figs
505.01 = 5.0501×10^2	5 sig. figs

3. Zeros that are at the right hand end of numbers are not significant, unless stated. For example:

80	1 sig. fig
800	1 sig. fig
8900	2 sig. figs
80090	4 sig. figs
990 = 9.9×10^2	2 sig. figs

4. Zeros to the left hand end of decimal numbers are not significant. For example:

0.8	1 sig. fig
0.148	3 sig. figs
0.0048	2 sig. figs
0.5048	4 sig. figs
0.0006 = 6×10^{-4}	1 sig. fig

5. Zeros at the right hand end of the decimal are significant. For example:

14.0	2 sig. figs
1.40	3 sig. figs
0.1480	4 sig. figs
1.4800	5 sig. figs
140.0 = 1.4×10^2	4 sig. figs

When recording measurements and significant figures, a result such as 6.54, which could be 6.54cm, and has a value with 3 significant figures. This implies that the first two digits may be accurate and the last digit is a close estimate. So the measurement of 6.54cm means that the 65mm are accurate but the 0.04mm is an estimate give or take 0.005mm.

3. Your Turn:

These practise examples will help to draw together your understanding about Scientific Notation and significant figures.

1. Write each of the following in scientific notation
2. State the number for significant figures in the value (you might like to refer to the rules).

a. 354

b. 18932

c. 0.0044

d. 0.000506

e. 960

f. 30000

g. 150.900

4. Power Operations

Powers are also called **indices**; we can work with the **indices** to simplify expressions and to solve problems.

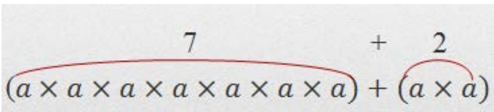
Some key ideas:

- Any base number raised to the power of 1 is the base itself: for example, $5^1 = 5$
- Any base number raised to the power of 0 equals 1, so: $4^0 = 1$
- Powers can be simplified if they are **multiplied** or **divided** and have the **same** base.
- Powers of powers are multiplied. Hence, $(2^3)^2 = 2^3 \times 2^3 = 2^6$
- A negative power indicates a reciprocal: $3^{-2} = \frac{1}{3^2}$

Certain rules apply and are often referred to as: **Index Laws.**

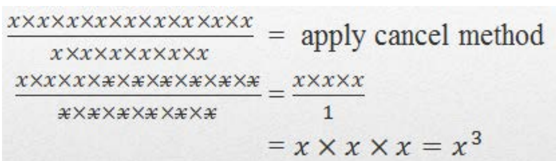
The first rule: $a^m \times a^n = a^{m+n}$

- To multiply powers of the same base, add the indices
- How does this work?
- Let's write out the 'terms'
- $a^7 \times a^2 = a^{7+2} = a^9$



The second rule: $\frac{x^9}{x^6} = x^{9-6} = x^3$

- To divide powers of the same base, subtract the indices
- How does this work?
- We are dividing so we can cancel
- Subtract 6 from 9 (numerator) and six from six (denominator)
- $\therefore \frac{x^9}{x^6} = x^{9-6} = x^3$
- From the second law we learn why x^0=1
- Any expression divided by itself equals 1
- so $\frac{x^3}{x^3} = 1$ or x^{3-3}=x^0 which is 1



The third rule: $(b^a)^m = b^{am}$

- To raise a power to a power, multiply the indices.
- $(b^2)^3$
- How does this work?
- $(b \times b) \times (b \times b) \times (b \times b) = b^6$
- Therefore, we multiply the indices.

The fourth rule: $(xy)^n = x^n y^n$

- A power of a product is the product of the powers.
- $(3 \times 4)^2$
- How does this work?
- $(\mathbf{3} \times \mathbf{4}) \times (\mathbf{3} \times \mathbf{4}) = \mathbf{3}^2 \times \mathbf{4}^2$
- $12 \times 12 = 9 \times 16 = (both\ equal\ 144)$
- Therefore, we can expand (remove the brackets).

The fifth rule: $({}^a/_b)^m = \frac{a^m}{b^m}$ (as long as *b* is not zero)

- A power of a quotient is the quotient of the powers.
- $({}^5/_2)^2 = \frac{5^2}{2^2}$How does this work?
- $(\mathbf{5} \div \mathbf{2}) \times (\mathbf{5} \div \mathbf{2}) = \mathbf{5}^2 \div \mathbf{2}^2$
- $2.5 \times 2.5 = \frac{25}{4} = (both\ equal\ 6\frac{1}{4})$

Therefore, we can expand (remove the brackets)

Fractional Indices: $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$

- This is related to the third index law and will be explained later in 'Fractional Powers'.

Below is a summary of the index rules:

Index rule	Substitute variables for values
$\boldsymbol{a^m \times a^n = a^{m+n}}$	$2^3 \times 2^2 = 2^{3+2} = 2^5 = 32$
$\boldsymbol{a^m \div a^n = a^{m-n}}$	$3^6 \div 3^3 = 3^{6-3} = 3^3 = 27$
$\boldsymbol{(a^m)^n = a^{mn}}$	$(4^2)^5 = 4^{2\times5} = 4^{10} = 1048\,576$
$\boldsymbol{(ab)^m = a^m b^m}$	$(2 \times 5)^2 = 2^2 \times 5^2 = 4 \times 25 = 100$
$\boldsymbol{({}^a/_b)^m = a^m \div b^m}$	$(10 \div 5)^3 = 2^3 = 8;\ (10^3 \div 5^3) = 1000 \div 125 = 8$
$\boldsymbol{a^{-m} = \frac{1}{a^m}}$	$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$
$\boldsymbol{a^{\frac{1}{m}} = \sqrt[m]{a}}$	$8^{1/3} = \sqrt[3]{8} = 2$
$\boldsymbol{a^0 = 1}$	$6^3 \div 6^3 = 6^{3-3} = 6^0 = 1; (6 \div 6 = 1)$

EXAMPLE PROBLEMS:

1. Simplify $6^5 \times 6^3 \div 6^2 \times 7^2 + 6^4 =$

$= 6^{5+3-2} \times 7^2 + 6^4$

$= 6^6 \times 7^2 + 6^4$

2. Simplify $g^5 \times h^4 \times g^{-1} =$

$= g^5 \times g^{-1} \times h^4$

$= g^4 \times h^4$

4. Your Turn

Apply the index laws/rules:

a. Simplify $5^2 \times 5^4 + 5^2 =$

b. Simplify $x^2 \times x^5 =$

c. Simplify $4^2 \times t^3 \div 4^2 =$

d. Simplify $(5^4)^3 =$

e. Simplify $\frac{2^4 3^6}{3^4} =$

f. Simplify $3^2 \times 3^{-5} =$

g. Simplify $\frac{9(x^2)^3}{3xy^2} =$

h. Simplify $a^{-1}\sqrt{a} =$ (applies to section 7)

What is the value of x for the following?

i. $49 = 7^x$

j. $\frac{1}{4} = 2^x$

k. $88 = 11^1 \times 2^x$

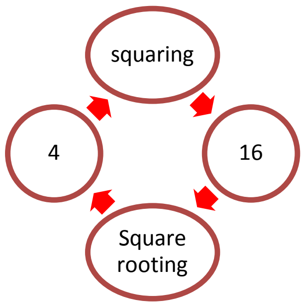
l. $480 = 2^x \times 3^1 \times 5^1$

m. Show that $\frac{16a^2b^3}{3a^3b} \div \frac{8b^2a}{9a^3b^5} = 6ab^5$

5. Roots

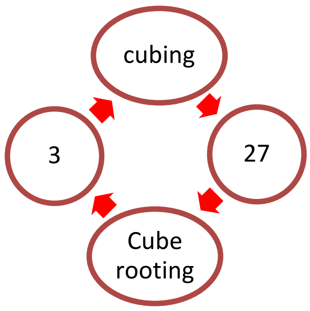
Previously we have looked at powers: $4^2 = 16$

- A root is used to find an unknown base: $\sqrt{16} = 4$
- The symbolic form $\sqrt{16}$ is expressed as, "What is the square root of 16?" or, "What number multiplied by itself equals 16?"
- Like exponents, the two most common roots ($\sqrt{\ }$ *and* $\sqrt[3]{\ }$) are called *square root* and *cube root*.
- $\sqrt{64}$ is expressed as the *square root* of 64. (Note square root does not have a 2 at the front, it is assumed)
- $\sqrt[3]{27}$ is expressed as the *cube root* of 27.



Let's look at the relationship between 3 and 27. We know that 3 cubed is 27 ($3 \times 3 \times 3$).

- So, 3 is the cube root of 27; $\sqrt[3]{27} = 3$ *and* $3^3 = 27$



Now let's look at negative numbers: $3^3 = 27$ *and* $(-3)^3 = -27$

- Yet, $5^2 = 25$ and $(-5)^2 = 25, \therefore\ -5$ is the square root of 25.
- Hence, $\sqrt{25} = \pm 5$

Simplifying roots is difficult without a calculator. This process requires an estimation of the root of a number. To estimate a root, it is helpful to know the common powers. Some common powers are listed below:

n^2		√	
$1^2 =$	1	$\sqrt{1} =$	1
$2^2 =$			
$3^2 =$			
$4^2 =$			
$5^2 =$			
$6^2 =$			
$7^2 =$			
$8^2 =$			
$9^2 =$	81	$\sqrt{81} =$	9
$10^2 =$			
$11^2 =$			
$12^2 =$			
$13^2 =$			

5. Your Turn:

Complete the table to the left.

Using the table:

$\sqrt{81} = 9$ (9 squared is 81)

$\sqrt{56}$ =? An example as such requires estimation. Look to the table, if the square root of 49 is 7, and the square root of 64 is 8, then the square root of 56 is between 7 and 8 (7.48).

A **surd** is a special root which cannot be simplified into a whole number. For instance, $\sqrt{4} = 2$, 2 is a whole number; therefore, $\sqrt{4}$ is not a surd. However, $\sqrt{3} = 1.732$, 1.732 is not a whole number; therefore, $\sqrt{3}$ is a surd. Large roots, such as $\sqrt{56}$ must be simplified to determine if they are surds. This process is explained on the next page.

5. Your Turn:

Which of the following are surds?

a. $\sqrt{1}$
b. $\sqrt{2}$
c. $\sqrt{3}$
d. $\sqrt{4}$
e. $\sqrt{9}$
f. $\sqrt[3]{2}$
g. $\sqrt[3]{8}$
h. $\sqrt[4]{16}$

6. Root Operations

Some basic rules:

- $\sqrt{x} \times \sqrt{y} = \sqrt{x \times y}$ for example, $\sqrt{6} \times \sqrt{4} = \sqrt{6 \times 4} = \sqrt{24}$
- $\sqrt{x^2} = x$ for example, $\sqrt{5^2} = 5$

- Simplifying Roots requires finding factors which are square numbers, such as the factors which are in the table on the previous page.

- If we look at the radical $\sqrt{56}$, it has multiple factors: 1×56 *or* 2×28 *or* 4×14 *or* 7×8.

  4×14 is key because it has a square number (4).

  Therefore, we can simplify $\sqrt{56} = \sqrt{4} \times \sqrt{14}$

  $$= 2\sqrt{14}$$

  The square root of four is two, thus we have 'two multiplied by the square root of 14'.

EXAMPLE PROBLEMS:

1. Simplify $\sqrt{32}$

   $\sqrt{32}$ 32 has the factors: 1 x 32 or 2 x 16 or 4 x 8. The biggest square number is 16.

   $$\therefore \sqrt{32} = \sqrt{16} \times \sqrt{2}$$
   $$= 4\sqrt{2}$$

2. Simplify $5\sqrt{18}$

   $5\sqrt{18}$ 18 has the factors: 1 x 18 or 2 x 9 or 3 x 6. The biggest square number is 9.

   $$\therefore 5\sqrt{18} = 5 \times \sqrt{9} \times \sqrt{2}$$
   $$= 5 \times 3 \times \sqrt{2}$$
   $$= 15\sqrt{2}$$

6. Your Turn:

Tip: To simplify the **radicals/surds** below, work out the factors of the number first. Work strategically by starting with $2 \times ?$. Work the factors until you get a squared number. It is important to select the **largest** factor that is a **squared** number.

a. Simplify $\sqrt{81}$

b. Simplify $\sqrt{72}$

c. Simplify $2\sqrt{48}$

d. Simplify $4\sqrt{338}$

7. Simplifying Fractions with Surds

When you are required to simplify radical expressions involving fractions, you are being asked to produce a single fraction without radicals in the denominator. This involves robust mathematical thinking and reasoning about core mathematical concepts. A tip is to start from what you know, recall what you have understood previously, and then apply that thinking systematically (step-by-step). Hence, to simplify fractions with radicals, think about what you know about fraction operations, such as converting fractions to make common denominators (to be able to add fractions). Then apply this thinking in a logical step-by-step manner.

Example Problems:

First example:

$\frac{\sqrt{3}}{\sqrt{2}} + \frac{2}{\sqrt{6}} =$ first we need to create the lowest common denominator to be able to add the fractions

$= \left(\frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{3}}{\sqrt{3}}\right) + \frac{2}{\sqrt{6}}$ To create a common denominator of $\sqrt{6}$, then we can add $\frac{2}{\sqrt{6}}$

$= \frac{3}{\sqrt{6}} + \frac{2}{\sqrt{6}}$ If we multiply two square root numbers we cancel the root, and then add $\frac{2}{\sqrt{6}}$

$= \frac{3+2}{\sqrt{6}} = \frac{5}{\sqrt{6}}$ Now we need to multiply both sides by $\sqrt{6}$ to remove the radical

$= \frac{5}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{5\sqrt{6}}{6}$

Second example:

$\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{10}} =$

$= \frac{1}{\sqrt{5}} \times \frac{\sqrt{2}}{\sqrt{2}} - \frac{1}{\sqrt{10}}$

$= \frac{\sqrt{2}}{\sqrt{10}} - \frac{1}{\sqrt{10}}$

$= \frac{\sqrt{2}-1}{\sqrt{10}}$

$= \frac{\sqrt{2}-1}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$

$= \frac{\sqrt{2\times 10}-1\times\sqrt{10}}{10}$

$= \frac{\sqrt{20}-\sqrt{10}}{10}$ Here we simplify $\sqrt{20} = \sqrt{5} \times \sqrt{4}, so\ 2\sqrt{5}$

$= \frac{2\sqrt{5}-\sqrt{10}}{10}$

7. Your Turn:

Have a go at these by following the steps above. It will take some deep mathematical thinking; however, if you approach it in a step-by-step manner you will feel the satisfaction of solving the 'puzzle'.

a. Simplify $\frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{2}}$

b. Simplify $\frac{1}{\sqrt{3}} - \frac{2\sqrt{3}}{\sqrt{15}}$

8. Fraction Powers/Exponents/Indices

Another way to express the cube root or square root is to use fraction powers. For example, an expression $9^{\frac{1}{2}}$ is also the same as the square root of nine: $\sqrt{9}$. Similarly, $\sqrt{16}$ could be expressed as $16^{\frac{1}{2}}$.

Then the same applies for the fractional power/exponent $\frac{1}{3}$, for example, $\sqrt[3]{8}$ can also be expressed as $8^{\frac{1}{3}}$

Thus we could say that an exponent such as $\frac{1}{x}$ implies to take the x th root.

Therefore, to generalise we could say that if $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$ then $x^{\frac{1}{n}} = \sqrt[n]{x}$

How does this work: Well let's look at $27^{\frac{1}{3}}$. What can we say about this expression?

- It is twenty seven to the fractional exponent of a third
- It implies the cube root of 27
- The cube root of 27 is 3. $(3 \times 3 \times 3) = 27$
- Thus we also know that $3^3 = 27$
- Now we could simplify $27^{\frac{1}{3}}$ to $(3^3)^{\frac{1}{3}}$
- So using our knowledge of index laws (page 21), we could also say that $(3^3)^{\frac{1}{3}}$ is the same as $3^{(3 \times \frac{1}{3})}$
- This then cancels out to $3^1 = 3$

This is a long way to explain how fractional powers/exponents work, but as with all mathematics, with practise a pattern forms making the 'how and why' of mathematical ideas more evident.

What about negative fractional powers/exponents?

Remember we worked with these briefly in section 15, where we looked at 6^{-3}, is the same as $\frac{1}{6^3}$: it is the reciprocal. Hence, we can generalise that $a^{-n} = \frac{1}{a^n}$

An example:

- $(27)^{-\frac{1}{3}}$
- If $a^{-n} = \frac{1}{a^n}$ then $(27)^{-\frac{1}{3}} = \frac{1}{27^{\frac{1}{3}}}$
- So we know that $27^{\frac{1}{3}} = \sqrt[3]{27}$
- $\therefore (27)^{-\frac{1}{3}} = \frac{1}{3}$

8. Your Turn:

Evaluate:

a. $(81)^{\frac{1}{2}}$

b. $(81)^{-\frac{1}{2}}$

9. Logarithms

With roots we tried to find the unknown base. Such as, $x^3 = 64$ is the same as $\sqrt[3]{64} = x$; (x is the base).

A **logarithm** is used to find an unknown **power/exponent**. For example, $4^x = 64$ is the same as $\log_4 64 = x$
This example above is spoken as: 'The logarithm of 64 with base 4 is x.' The base is written in subscript.

The general rule is: $N = b^x \Leftrightarrow \log_b N = x$

- In mathematics the base can be any number, but only two types are commonly used:
  - $\log_{10} N$ $(base\ 10)$ is often abbreviated as simply Log, and
  - $\log_e N$ $(base\ e)$ is often abbreviated as Ln or natural log
- $\log_{10} N$ is easy to understand for: $\log_{10} 1000 = \log 1000 = 3$ $(10^3 = 1000)$
  $\log 100 = 2$
- Numbers which are not 10, 100, 1000 and so on are not so easy. For instance, $\log 500 = 2.7$ It is more efficient to use the calculator for these types of expressions.

9. Your Turn:

Write the logarithmic form for:

a. $5^2 = 25$
b. $6^2 = 36$
c. $3^5 = 243$

Solve and write in exponential form:

d. $\log_2 8 =$
e. $\log_3 27 =$
f. $\log_3 81 =$
g. $\log_4 64 =$

Use your calculator to solve

h. Log 10000 =

i. Log 350 =

j. Ln 56 =

k. Ln 100 =

Let's explore further

Logs to the base 10 are used in chemistry where very small concentrations are involved. For instance, acidity is determined by the concentration of H^+ ions (measured in moles/litre and written as $[H^+]$ in a solution. In pure water $[H^+]$ = 0.0000001 mole/litre of 1×10^{-7}

$[H^+]$ = 10^{-14} 10^{-13} 10^{-12} 10^{-11} 10^{-10} 10^{-9} 10^{-8} 10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1}

← less acidic | ↑ neutral | More acidic →

A special scale called pH has been developed to measure acidity and it is simply the 'negative index' of the above scale. You may have noticed this scale if you help out with the balancing a swimming pool to ensure the water is safe for swimming.

pH = 14 13 12 11 10 9 8 7 6 5 4 3 2 1

← less acidic | ↑ neutral | More acidic →

So a solution with a pH=7 is neutral, while pH =3 is acidic, pH=2 is more acidic, pH 12 is alkaline and so on.

8. Your Turn:

Use the scales above to answer these questions

Convert these $[H^+]$ concentrations to scientific notation and then pH.

l. $[H^+]$ = 0.001 moles/litre

m. $[H^+]$ = 0.00000001 moles/litre

n. $[H^+]$ = 0.0000001 moles/litre

o. $[H^+]$ = 0.000000000001 moles/litre

p. When converting from the $[H^+]$ scale to the pH scale, we take "the negative of the log to base 10 of the $[H^+]$." Can you translate this English statement into a mathematical equation?

10. Helpful Websites

Powers: http://www.mathsteacher.com.au/year8/ch07_indices/01_ind/ind.htm

Roots: http://www.math.utah.edu/online/1010/radicals/

Logarithms: http://www.mathsisfun.com/algebra/logarithms.html

11. Answers

1.

a. $2^3 = 8$
b. $9^2 = 81$
c. $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
d. $(-4)^2 = 16$
e. $(-3)^3 = -27$
f. $(-\frac{1}{4})^2 = \frac{1}{16}$
g. $0^3 = 0$
h. $(-0.1)^2 = 0.01$
i. $(0.5)^3$ =0.125 as is **c.** above

j. $4^{-2} = \frac{1}{16}$
k. 1^{-11} =1
l. $4^{-1} = \frac{1}{4}$
m. $(-4)^{-1} = -\frac{1}{4}$
n. $(0.5)^{-4}$= $(\frac{1}{2})^{-4}$ $= 2^4$ =16
o. $(\frac{3}{4})^{-3} = \frac{4^3}{3^3} = \frac{64}{27}$
p. $(-2)^{-3} = -\frac{1}{8}$

2. Write the following in scientific notation:

a. $450 = 4.5 \times 10^2$

b. $90000000 = 9.0 \times 10^7$

c. 3.5 = is already in standard form

d. $0.0975 = 9.75 \times 10^{-2}$

Write the following numbers out in full:

e. $3.75 \times 10^2 = 375$

f. $3.97 \times 10^1 = 39.7$

g. $1.875 \times 10^{-1} = 0.1875$

h. $-8.75 \times 10^{-3} = -0.00875$

3.

a. 354	3.54×10^2	3 sig. figs
b. 18932	1.8932×10^4	5 sig. figs
c. 0.0044	4.4×10^{-3}	2 sig. figs
d. 0.000506	5.06×10^{-4}	3 sig. figs
e. 960	9.6×10^2	2 sig. figs
f. 30000	3×10^4	1 sig. fig
g. 150.900	1.509×10^2	6 sig. figs

4. Simplify

a. $5^2 \times 5^4 + 5^2 = 5^6 + 5^2$
b. $x^2 \times x^5 = x^7$
c. $4^2 \times t^3 \div 4^2 = t^3$
d. $(5^4)^3 = 5^{12}$
e. $\frac{2^4 3^6}{3^4} = 2^4 3^2 = 16 \times 9 = 144$

f. $3^2 \times 3^{-5} = 3^{-3} = \frac{1}{27}$
g. $\frac{9(x^2)^3}{3xy^2} = \frac{9x^6}{3xy^2} = \frac{3x^5}{y^2}$
h. $a^{-1}\sqrt{a} = a^{-1} \times a^{\frac{1}{2}} = a^{-\frac{1}{2}}$
$= \frac{1}{\sqrt{a}}$ $or \frac{1}{a^{\frac{1}{2}}}$

i. $x = 2$

j. $x = -2$

k. $x = 3$

l. $x = 5$

m. Show that $\frac{16a^2b^3}{3a^3b} \div \frac{8b^2a}{9a^3b^5} = 6ab^5$ $\quad \frac{\cancel{16}a^2b^3}{\cancel{3}a^3b} \times \frac{\cancel{9}a^3b^5}{\cancel{8}b^2a}$

$= \frac{2a^5b^8 \times 3}{b^3a^4} = \frac{2a^1b^5 \times 3}{1} = 6ab^5$

5.

n^2		√	
1^2	1	$\sqrt{1}$	1
2^2	4	$\sqrt{4}$	2
3^2	9	$\sqrt{9}$	3
4^2	16	$\sqrt{16}$	4
5^2	25	$\sqrt{25}$	5
6^2	36	$\sqrt{36}$	6
7^2	49	$\sqrt{49}$	7
8^2	64	$\sqrt{64}$	8
9^2	81	$\sqrt{81}$	9
10^2	100	$\sqrt{100}$	10
11^2	121	$\sqrt{121}$	11
12^2	144	$\sqrt{144}$	12
13^2	169	$\sqrt{169}$	13

The following are surds: $\sqrt{2}\ \sqrt{3}\ \sqrt[3]{2}$

6. Simplify

a. $\sqrt{81} = 9$

b. $\sqrt{72}\ factors\ are\ 2 \times 36\ so\ \sqrt{2} \times \sqrt{36} = \sqrt{2} \times 6 = 6\sqrt{2}$

c. $2\sqrt{48}(factors\ 3 \times 16)\ so\ 2 \times \sqrt{3} \times \sqrt{16} = 2 \times 4 \times \sqrt{3} = 8\sqrt{3}$

d. $4\sqrt{338}(factors\ 169 \times 2)\ so\ 4 \times \sqrt{169} \times \sqrt{2} = 4 \times 13 \times \sqrt{2} = 52\sqrt{2}$

7. a.

$$\frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{2+3}{\sqrt{6}} = \frac{5}{\sqrt{6}}$$
$$= \frac{5}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{6}$$

i.

b.

$$\frac{1}{\sqrt{3}} - \frac{2\sqrt{3}}{\sqrt{15}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{5}}{\sqrt{5}} - \frac{2\sqrt{3}}{\sqrt{15}}$$
$$= \frac{\sqrt{5}-2\sqrt{3}}{\sqrt{15}}$$
$$= \frac{\sqrt{5}-2\sqrt{3}}{\sqrt{15}} \times \frac{\sqrt{15}}{\sqrt{15}}$$
$$= \frac{\sqrt{75}-2\sqrt{45}}{15}$$
$$= \frac{\sqrt{25}\times\sqrt{3}-2\sqrt{9}\times\sqrt{5}}{15} = \frac{5\sqrt{3}-6\sqrt{5}}{15}$$

8. a. $\sqrt{81} = \pm 9$ b. $\frac{1}{\sqrt{81}} = \frac{1}{\pm 9} = \pm\frac{1}{9}$

9. Write the logarithmic form for:

a. $\log_5 25 = 5$
b. $\log_6 36 = 2$
c. $\log_3 243 = 5$

Solve and write in exponential form:

d. $\log_2 8 = 3$ equivalent to $2^3 = 8$
e. $\log_3 27 = 3$ equivalent to $3^3 = 27$
f. $\log_3 81 = 4$ equivalent to $3^4 = 81$
g. $\log_4 64 = 3$ equivalent to $4^3 = 64$
h. 4
i. 2.54
j. 4.03
k. 4.61
l. $[H^+]$ = 0.001 moles/litre = 10^{-3} pH=3
m. $[H^+]$ = 0.00000001 moles/litre = 10^{-8} pH=8
n. $[H^+]$ = 0.0000001 moles/litre = 10^{-7} pH=7
o. $[H^+]$ = 0.000000000001 moles/litre = 10^{-12} pH12
p. pH= $-\log_{10} [H^+]$