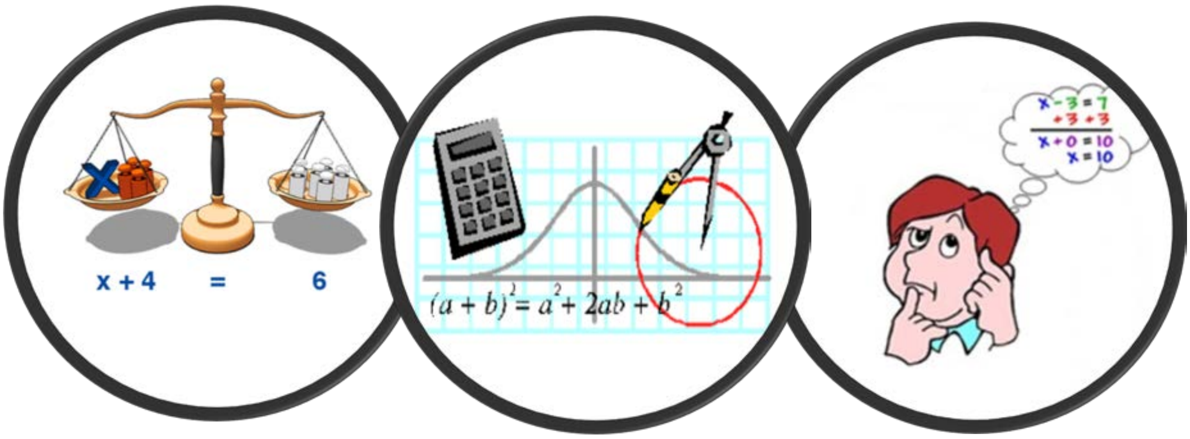
Maths Module 6

Algebra Solving Equations

This module covers concepts such as:

- solving equations with variables on both sides
- multiples and factors
- factorising and expanding
- function notation



www.jcu.edu.au/students/learning-centre

TheLearningCentre
UNLOCK YOUR POTENTIAL

Module 6

Solving Equations

1. Solving Equations

An **equation** states that two quantities are equal. This means that the left hand and right hand sides of the equals sign are equivalent, they balance. The equation may contain an **unknown quantity** that we wish to find. In the equation, $5x + 10 = 20$, the unknown quantity is x. This means that 5 multiplied by something that is then added to 10, will equal 20.

- To solve an equation means to find all values of the unknown quantity so that they can be substituted to make the left side **equal** to the right side and vice versa.
- Each such value is called a **solution**, or alternatively a **root** of the equation. In the example above, the solution is $x = 2$ because when 2 is substituted, both the left side and the right side equal $20 \therefore$ the sides balance. The value $x = 2$ is said to satisfy the equation.
- Sometimes we are required to rearrange or transpose the equation to solve it: essentially what we do to one side we do to the other.

(Croft & Davison, 2010, p. 109)

- Hence, as above $5x + 10 = 20$, we first rearrange by subtracting 10 from the LHS and the RHS.
  - $5x + 10\,(-10) = 20(-10); 5x = 10$
  - Then we rearrange further to get the $\boldsymbol{x}$ **on its own**; so we divide both LHS and RHS by 5.
  - $\frac{5x}{5} = \frac{10}{5}$; now we have a solution: $x = 2$
  - To check if this is correct we substitute $x\ for\ 2$; $(5 \times 2) + 10 = 20 \therefore 10 + 10 = 20$✓

This was a two-step equation – which will be covered later.

Four principles to apply when solving an equation:

1. **Work towards the variable:** Our aim is to get the variable by itself on one side of the equation. (So for the above we would aim to get the x by itself on one side)

$$(\boldsymbol{x} =)$$

2. **Use the opposite mathematical operation**: thus to remove a constant or coefficient, we do the opposite on both sides:

Opposite of $\times\ \boldsymbol{is}\ \div$

Opposite of $+\ \boldsymbol{is}\ -$

Opposite of $\boldsymbol{x^2}\ \boldsymbol{is}\ \sqrt{\boldsymbol{x}}$

3. **Maintain balance**: "What we do to one side, we must do to the other side of the equation."

4. **Check:** Substitute the value back into the equation to see if the solution is correct.

One-step Equations

ADDITION EXAMPLE: $\boldsymbol{x + (-5) = 8}$

Step 1: The constant, (-5), is the first target. So we need to do the opposite of plus (-5) which is to subtract $(-5\ )$ from the LHS and the RHS:

$x + (-5) - (-5) = 8 - (-5)$

$x = 8 + 5$ (Here we apply our knowledge of negative and positive integers from Module 1.)

$\therefore x = 13$

Check $13 + (-5) = 8$

$13 - 5 = 8$ ✓

SUBTRACTION EXAMPLE: $x - 6 = (-4)$

Step 1: The constant, (-6) is the target. The opposite of subtract 6 is to add 6.

$x - 6 + 6 = (-4) + 6$

So $x = (-4) + 6$

$\therefore x = 2$

Check $2 - 6 = (-4)$ ✓

1. Your Turn:

Solve for x:

a. $x + 6 - 3 = 18$

b. $7 = x + (-9)$

c. $x - 12 = (-3)$

d. $18 - x = 10 + (-6)$

Two-step Equations

The following equations require two steps to single out the variable.

ADDITION EXAMPLE: $\boldsymbol{2x + 6 = 14}$

In words: What number can we double then add six so we have a total of fourteen?

Step 1: The **constant,** 6, is our first target. If we subtract 6 from both sides, we create the following equation:

$2x + 6 - 6 = 14 - 6$ (The opposite of $+6$ is -6)

$2x = 8$ $(+6 - 6 = 0)$

Step 2: The only number left on the same side as the variable is the **coefficient**, 2. It is our second target. If we divide both sides by two, we create the following equation: (Note: between the 2 and the x is an invisible multiplication sign): $\frac{2x}{2} = \frac{8}{2}$ (The opposite of $2x$ is $\div 2$)

$\therefore x = 4$

Check. If we substitute 4 into the equation we have: $2 \times 4 + 6 = 14$

$8 + 6 = 14$

$14 = 14$ ✓ (We are correct)

Subtraction example:

Solve for j: $3j - 5 = 16$

Step 1: $3j - 5 = 16$ (The first target is 5)

$3j - 5 + 5 = 16 + 5$ (Opposite of -5 is $+5$)

Thus, $3j = 21$

Step 2: The second target is 3 and the opposite of $\times 3$ is $\div 3$)

$\frac{3j}{3} = \frac{21}{3}$

$\therefore\ j = 7$

Check: $3 \times 7 - 5 = 16$ ✓

Multi-step example:

Solve for T: $\mathbf{\frac{3T}{12} - 7 = 6}$

Set your work out in logical clear to follow steps

$\mathbf{\frac{3T}{12} - 7 = 6}$	(Target 7 then 12 then 3)
$\frac{3T}{12} - 7 + 7 = 6 + 7$	(add 7 to LHS and RHS)
$\frac{3T}{12} = 13$	(now target 12 to get $3T$ on its own)
$\frac{3T}{12} \times 12 = 13 \times 12$	(multiply LHS and RHS by 12)
$3T = 156$	(now target 3 to get the T on its own)
$\frac{3T}{3} = \frac{156}{3}$	(divide LHS and RHS by 3)
$\therefore T = 52$	

Check: $(3 \times 52) \div 12 - 7 = 6$ ✓

1. Your Turn:

Solve the following:

e. $5x + 9 = 44$

f. $\frac{x}{9} + 12 = 30$

g. $3y + 13 = 49$

h. $4x - 10 = 42$

i. $\frac{x}{11} + 16 = 30$

2. Solving Equations with Fractions

So far we have looked at solving one and two step equations. The last example had fractions too, which we will explore more deeply in this section. First, let us go back and revise **terms**. In Module 5 we covered terms, but it is important to remember that in algebra, terms are separated by a plus $(+)$ or minus $(-)$ sign or by an equals $(=)$ sign. Whereas variables that are multiplied or divided are considered **one** term.

For example, there are four terms in this equation: $\mathbf{8ab + 3ab + 4 = 70}$

and also: $\mathbf{3(x+2) + 8 - 6 = 20}$

When working with fractions, it is generally easier to eliminate the fractions first, as follows:

EXAMPLE ONE: Let's solve for : $\mathbf{\frac{2}{5}x - 6 = 4}$ (There are 3 terms in this equation?)

Step 1: Eliminate the fraction, to do this we work with the denominator first, rather than multiply by the reciprocal as this can get messy. So both sides of the equation will be multiplied by 5, which includes distributing 5 through the brackets :

$$5(\frac{2}{5}x - 6) = 5 \times 4$$
$$2x - 30 = 20$$

Step 2: Target the constant: $2x - 30 + 30 = 20 + 30$ (The opposite of subtraction is addition)

$$2x = 50$$

Step 3: Target the variable: $\frac{2x}{2} = \frac{50}{2} \therefore x = 25$

Check: $(\mathbf{\frac{2}{5} \times \frac{25}{1}) - 6 = 4}$

$(2 \times 5) - 6 = 4 \therefore 10 - 6 = 4$✓

EXAMPLE TWO: Let's solve for x: $\frac{3}{5}(x+5) = 9$

Step 1: Eliminate the fraction: Again we work with the denominator and multiply everything by 5:

$5[\frac{3}{5}(x+5)] = 5(9)$ Note: the 5 is not distrbuted through the brackets $(x+5)$because $\frac{3}{5}(x+5)$ is **one** term.

The result: $3(x+5) = 45$

Step 2: Target the variable by dividing by 3: $\frac{3(x+5)}{3} = \frac{45}{3}$; $x + 5 = 15$

Now we target the 5 : $x + 5 - 5 = 15 - 5 \quad \therefore x = 10$

Check: : $\frac{3}{5}(10+5) = 9$; $\frac{3}{5} \times \frac{15}{1} = 9$✓

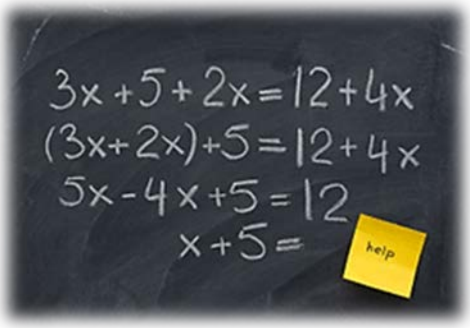
2. Your Turn: solve for x :

a. $\frac{1}{2}x + \frac{3}{2}(x-4) = 6$ (Tip: 3 terms)

b. $4x + \frac{1}{2}(2x-4) = 18$

3. Solving Equations with Variables on Both Sides

Solving equations with variables on both sides can be difficult and requires some methodical mathematical thinking. Remembering that both sides are equivalent, the goal is to get all of the constants on one side of the equation and the variables on the other side of the equation. As we have explored previously, it is helpful to deal with the fractions first.



We can begin by solving the equation on the front cover of this module: This image was accessed from a website you may find resourceful: http://www.algebra-class.com

Example One:

Solve for : $3x + 5 + 2x = 12 + 4x$

Work towards rearranging the equation to get all of the constants on the RHS and the variables on the LHS:

$3x + 5 + 2x - 4x = 12 + 4x - 4x$ Let's start by subtracting $4x$ from both sides.

The result: $3x + 5 + 2x - 4x = 12$

$3x + 5 - 5 + 2x - 4x = 12 - 5$ Now subtract 5 from both sides.

$3x + 2x - 4x = 7$ Now simplify by collecting like terms. $(3 + 2 - 4 = 1)$

$1x = 7 \therefore x = 7$

Check: $3x + 5 + 2x = 12 + 4x$

$21 + 5 + 14 = 12 + 28$ (Substitute the variable for 7)

$40 = 40$✓

Example Two:

Solve for x: $6x + 3(x + 2) = 6(x + 3)$

Again, work towards moving all the constants on one side and the variables on the other. First we could distribute through the brackets which will enable us to collect like terms:

$6x + 3x + 6 = 6x + 18$ (Distribute through the brackets)

$6x + 3x + 6 - 6 = 6x + 18 - 6$ (Subtract 6 from both sides first, to get constants on RHS)

The result: $6x + 3x = 6x + 18 - 6$

$9x - 6x = 6x - 6x + 18 - 6$ (Subtract $6x$ from both sides to get variables on LHS)

$3x = 12 \quad \therefore \quad x = 4$

Check: $6x + 3(x + 2) = 6(x + 3); 24 + 18 = 6 \times 7 \quad \therefore \quad 42 = 42$✓

3. Your Turn:

Transpose to make x the subject:

a. $y = 3x$

b. $y = \frac{1}{x}$

c. $y = 7x - 5$

d. $y = \frac{1}{2}x - 7$

e. $y = \frac{1}{2x}$

Solve for x:

(f. - m. adapted from Muschala et al. (2011, p. 99)

f. $9x = 5x + 16$

g. $12x + 85 = 7x$

h. $-8x = -13x - 65$

i. $59 + x = 2 - 2x$

j. $3(x+10)=2x$

k. $2(2x-1)=6(x+2)$

l. $-18+x=-x+12$

m. $3(x-1)=2(x+5)$

n. $\frac{3}{4}x+6=26-(\frac{1}{2}x+10)$

o. $2x+8=\frac{1}{2}(43+x)$

p. $\frac{3}{4}(x+81)=(4x-1)$

4. Multiples and Factors

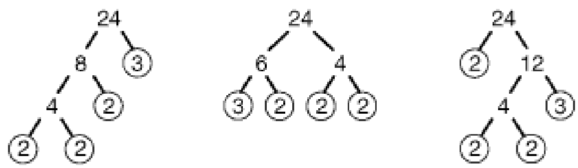
This section will provide you with a refresh on factors and multiples. These concepts relate to understanding fractions, ratios and percentages, and they are integral to factoring when solving more difficult problems.

Multiples, common multiples and the LCM

- A **multiple** of a given a number, into which that number can be divided exactly.
  For example: multiples of 5 are 5, 10, 15, 20, 25, 30…
- A **common multiple** is a multiple in which two or more numbers have in common.
  For example: 3 and 5 have multiples in common 15, 30, 45…
- The **lowest common multiple** LCM is the lowest multiple that two numbers have in common.
  For example: 15 is the LCM of 3 and 5

Factors, common factors and the HCF

- **Factor** – a whole number that can be multiplied a certain number of times to reach a given number.
  3 is factor of 15 because it can be multiplied by 5 to get 15
- **A common factor** is a factor that two or more numbers have in common.
  3 is a common factor of 12 and 15
- The **highest common factor – HCF**
  What is the HCF of 20 and 18?
  We list all of the factors first.
  The factors of 20: (1, 2, 4, 5, 10, 20)
  The factors of 18: (1, 2, 3, 6, 9, 18)
  The common factors are 1 and 2 and the **highest** common factor is 2
- **Proper factors:** All the factors apart from the number itself
  The factors of 18 (1, 2, 3, 6, 9, 18)
  Thus the proper factors of 18 are 1, 2, 3, 6, 9
- **Prime number**: Any whole number greater than zero that has exactly two factors – itself and one
  2, 3, 5, 7, 11, 13, 17, 19…
- **Composite number:** Any whole number that has more than two factors
  4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20…
- **Prime factor**
  A factor that is also a prime number
- **Factor tree**
  A tree that shows the prime factors of a number



As the factor trees show, some numbers have many factors. This is a helpful way to find the factors of a given number. A more effective method for clarity when thinking and reasoning mathematically is to list all of the factors.

4. Your Turn:

List all of the factors of each of the following numbers:

a. 14____

b. 134_______________________________________

c. 56____

d. 27____

e. 122_______________________________________

5. Factorising and Expanding

The distributive law is commonly used in algebra. It is often referred to as either 'expanding the brackets' or 'removing the brackets'.

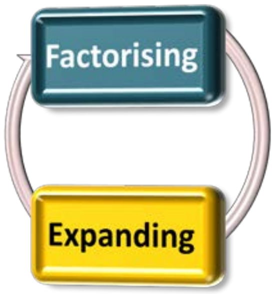
- **Expanding** involves taking what is outside of the brackets and moving it **through** the brackets. For example:

  $3(x+2)$ can be expanded to $3 \times x + 3 \times 2$ which is $3x+6$
  This means that: $3(x+2) = 3x+6$

- Working in the opposite direction is called **factorisation** and this process is slightly more difficult.

- Factorisation requires finding the highest common factor. (The HCF then sits outside of the brackets.)

- Let's look at $3x+6$, we can factorise this back again from the expanded form.
  - First we need to find a common 'factor'; we can see that x can be divided by 3 and 6 can be divided by 3. Our common factor is 3. Now we put this 3 outside of the brackets (because we have divided each of the terms by three) to be multiplied by everything inside of the brackets.
  - So we have $3(x+2)$
- Another factorisation example:
  - $5x + 15x^2 - 30x^3$ has three terms $5x$ and $15x^2$ and $30x^3$
  - All three terms contain x and can be divided by 5 ∴ $5x$ is a common factor.
  - Next we divide each term by $5x$.
  - $5x \div 5x = 1; 15x^2 \div 5x = 3x;$ and $30x^3 \div 5x = 6x^2$
  - We end up with: $5x(1+3x-6x^2)$.



- Now we can do the reverse and practise expanding the brackets.
  - So expand $5x(1+3x-6x^2)$
  - Multiply each term inside the brackets by $5x$
  - $5x \times 1 = 5x; 5x \times 3x = 5 \times 3 \times x \times x = 15x^2;$ and $5x \times 6x^2 = 5 \times 6 \times x \times x \times x = 30x^3$
  - Now we have expanded the brackets to get $5x + 15x^2 - 30x^3$

EXAMPLES:

Remove the Brackets:

- $5x(2+y)$ expands to $(5x \times 2) + (5x \times y)$ which is simplified to $10x + 5xy$
- $-x(2x+6)$ expands to $(-x \times 2x) + (-x \times 6)$ which is simplified to $-2x^2 - 6x$

Factorise:

- $3x + 9x - x^2$ has only the variable x in common ∴ $x(3+9-x)$
- $12x^3 + 4x^2 - 20x^4$ has $4x^2$ as the highest common factor ∴ $4x^2(3x+1-5x^2)$

5. Your Turn:

Expand the first two and then factorise back again, then factorise the next two and then expand for practise:

a. Expand $2y^2(3x+7y+2)$

b. Expand $-1(3x+2)$

c. Factorise $3x + 6x^2 - 9x$

d. Factorise $24x + 42xy - 60x^3$

6. Function Notation

Most of the equations we have worked with so far have included a single variable. For example, $2x + 5 = 11$, x is the single variable. Single variable equations can be solved easily. Yet, in real life you will often see questions that have two variables:

The cost of fuel is equal to the amount of fuel purchased and the price per litre \$1.50/L.

The mathematical equation for this situation would be:

$Cost = Amount \times PPL$ (price per litre)

$\therefore C = A \times 1.5$

If the amount of fuel is 20 litres, the cost can be calculated by using this formula:

$C = 20 \times 1.5$

$\therefore C = 30$

Another way to describe the above scenario is to use **function notation**. The cost is a **function** of amount of fuel.

If we let x represent the amount, then the cost of fuel can be represented as $f(x)$

Therefore: $f(x) = 1.5x$. We can now replace x with any value.

If $f(x) = 1.5x$ then solve for $f(3)$

This means replace x with the number 3.

$f(3) = 1.5 \times 3$

$\therefore f(3) = 4.5$

Example Problems:

1. $f(x) = 5x + 7$ Solve for $f(4)$ and $f(-2)$

$f(4) = 5 \times 4 + 7$ $\qquad$ $f(-2) = 5 \times -2 + 7$

$\therefore f(4) = 27$ $\qquad$ $\therefore f(-2) = -3$

2. $f(x) = \frac{3}{x} + x^2$ Solve for $f(6)$ *and* $f(-9)$

$$f(6) = \frac{3}{6} + 6^2 \qquad f(-9) = \frac{3}{-9} + (-9)^2$$

$$\therefore f(6) = 36.5 \qquad \therefore f(-9) = 80\frac{2}{3}$$

6. Your Turn:

a. $f(x) = 6x + 9$ Solve for $f(4)$ and $f(0)$

b. $f(x) = x^2 + \frac{6}{x}$ Solve for $f(6)$ and $f(-6)$

7. Answers

1. a. $x = 15$ b. $x = 16$ c. $x = 9$ d. $x = 14$ e. $x = 7$ f. $x = 162$ g. . $y = 12$

   h. $x = 13$ i. $x = 154$

2. a. $\frac{1}{2}x + \frac{3}{2}(x - 4) = 6$
   $2\left(\frac{1}{2}x\right) + 2\left[\frac{3}{2}(x - 4)\right] = 2(6)$
   $1x + 3(x - 4) = 12$
   $1x + 3x - 12 = 12$
   $4x - 12 = 12$
   $4x = 24 \therefore x = 6$

   b. $4x + \frac{1}{2}(2x - 4) = 18$
   $2(4x) + 2[\frac{1}{2}(2x - 4)] = 2(18)$
   $8x + 2x - 4 = 36$
   $10x - 4 = 36$
   $10x = 40; \therefore x = 4$

3. a. $\frac{y}{3} = x$ b. $x = \frac{1}{y}$ c. $\frac{y+5}{7} = x$ d. $2y + 14 = x$ e. $x = \frac{1}{2y}$ f. $x = 4$ g. $x = -17$ h. $x = -13$
   i. $x = -19$ j. $x = -30$ k. $x = -7$ l. $x = 15$ m. $x = 13$ n. $x = 8$ o. $x = 9$ p. $x = 19$

4. a. 1, 2, 7, 14 b. 1, 2, 67, 134 c. 1, 2, 4, 7, 8, 14, 28, 56 d. 1, 3, 9, 27 e. 1, 2, 61, 122

5. a. $6xy^2 + 14y^3 + 4y^2$ b. $-3x - 2$ c. $6x(x - 1)$ d. $6x(4 + 7y - 10x^2)$

6. a. f(4) = 33; f(0) = 9 b. f(6) = 37; f(-6) = 35

8. Helpful Websites

Like Terms: http://www.freemathhelp.com/combining-like-terms.html

Solving Equations: http://www.purplemath.com/modules/solvelin.htm

Solving Equations: http://www.algebra-class.com/solving-algebra-equations.html

Expanding and Factorising: http://www.mathsrevision.net/gcse/pages.php?page=14

Function Notation: http://www.purplemath.com/modules/fcnnot.htm