Maths Refresher

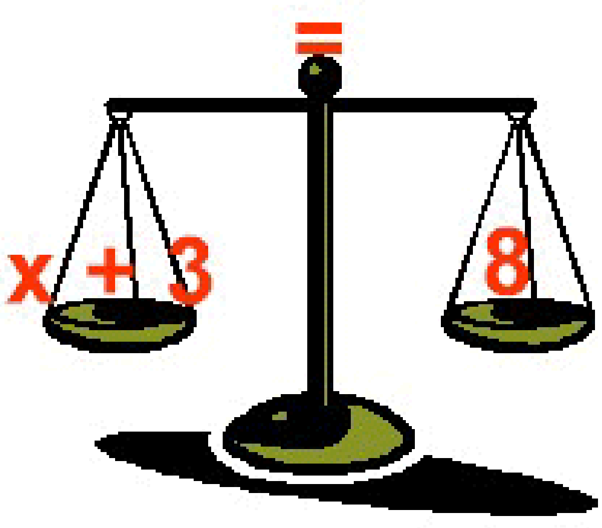
Expanding and Factorising

Learning, Teaching
and Student Engagement

Expanding and Factorising

Learning intentions ….

- Recap
- Expanding equations
- Factorising equations
- Identity: perfect pairs
- Difference of two squares



Introduction

- Algebra requires you to **manipulate** algebraic expressions
- We have covered simplifying expressions and solving equations
- Now we look at manipulating expressions through expanding and factorising
- First, we recap some mathematical ideas that will assist factorisation
- Second, we revise the distributive law
- Third, you will learn how to expand two or more sets of brackets

Recap

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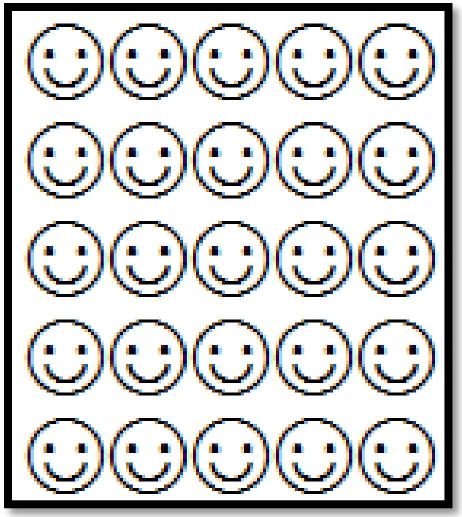
Multiples

A **multiple** is a number that can be divided into a given number exactly

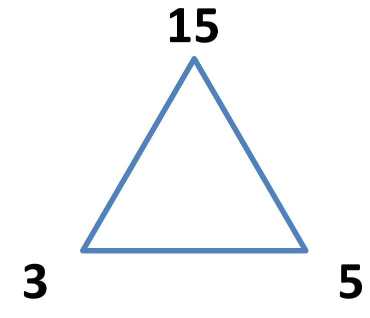
- For example: multiples of 5 are 5, 10, 15, 20, 25…



Recap

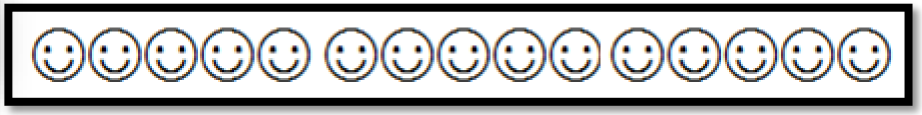
- **Common multiples and the LCM**

- A **common multiple** is a multiple in which two or more numbers have in common
  - For example: 3 and 5 have multiples in common 15, 30, 45…
- The **lowest common multiple** LCM is the lowest multiple that two numbers have in common
  - For example: 15 is the LCM of 3 and 5

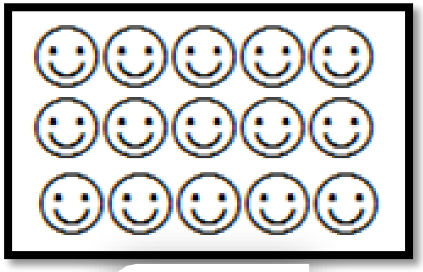


Recap

- **Factor** – a whole number that can be multiplied a certain number of times to reach a given number
  - 3 is factor of 15 and 15 is a multiple of 3
  - The other factors are 1and 15

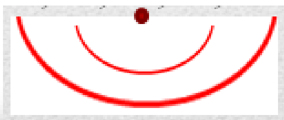


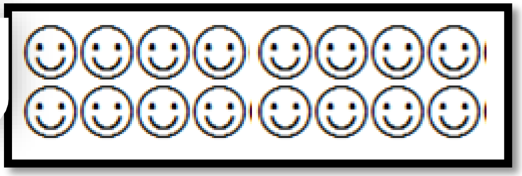
1×15



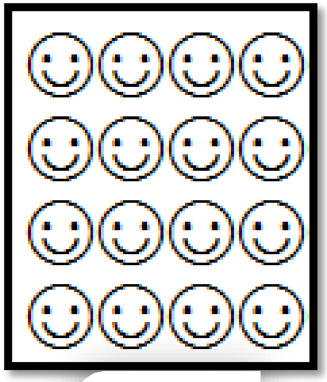
3×5

  - 4 is a factor of 16 and 16 is a multiple of four, other factors
  - 1, 2, 4, 8, 16

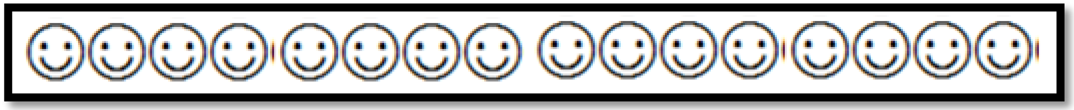




2×8



4×4



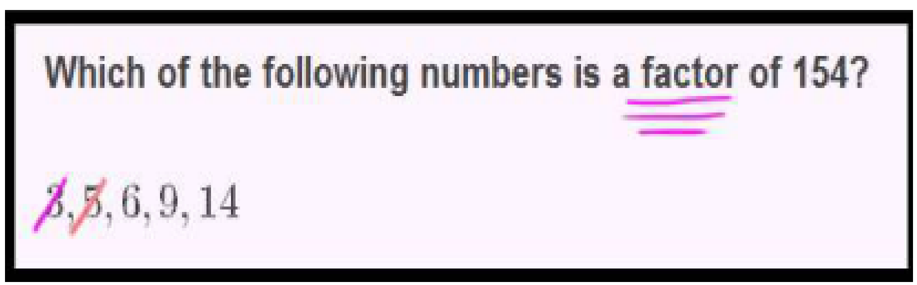
1×16

Recap

- **A common factor** is a common factor that two or more numbers have in common
  - – 3 is a common factor of 12 and 15
  - – 5 is a common factor of 15 and 25
  - – 6 is a common factor of 12 and 18
- The **highest common factor – HCF**
  - – What is the HCF of 20 and 18?
  - – the factors of 20 (1, 2, 4, 5, 10, 20) and 18 (1, 2, 3, 6, 9, 18)
  - – the common factors are 1 and 2 and
  - – the HCF is 2

Recap

- We could say that a number is a factor of given number if it is a multiple of that number
- For example,
  - 9 is a factor of 27 and 27 is a multiple of 9
  - 7 is a factor of 35 and 35 is a multiple of 7
  - 14 is a factor of 154 and 154 is a multiple of 14



[YouTube clip](https://www.khanacademy.org/math/pre-algebra/factors-multiples/divisibility_and_factors/v/finding-factors-and-multiples)

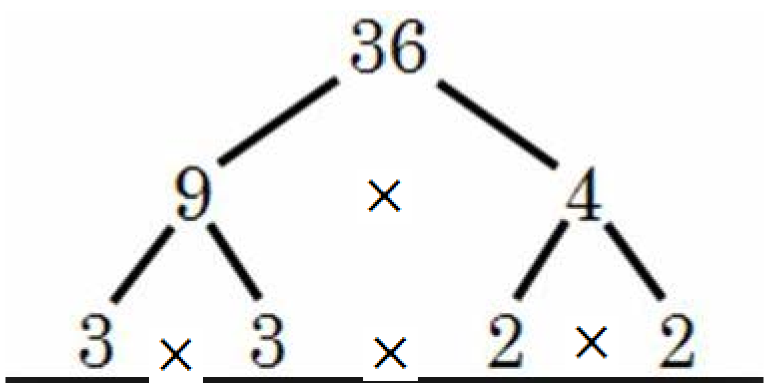
- https://www.khanacademy.org/math/pre-algebra/factors-multiples/divisibility_and_factors/v/finding-factors-and-multiples

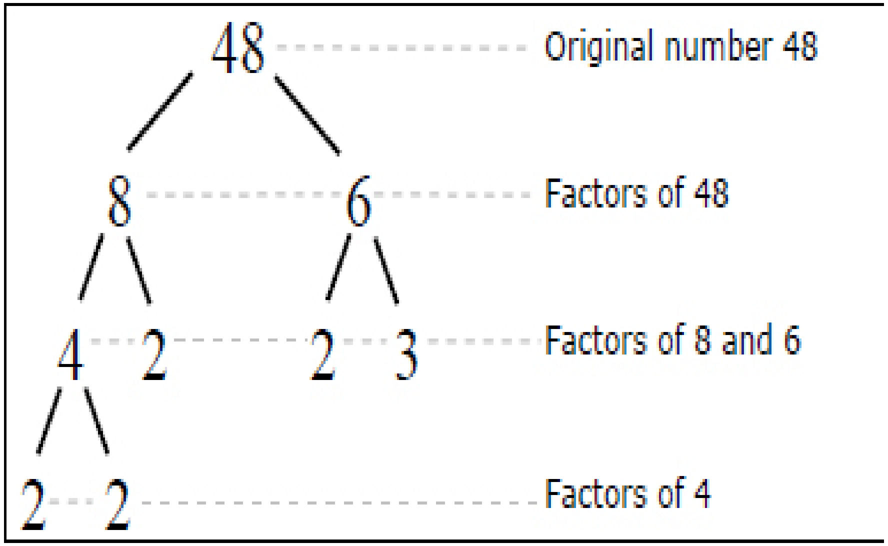
Recap

- **Proper factors**
  - All the factors apart from the number itself
  - 18 (1, 2, 3, 6, 9, 18) 1, 2, 3, 6, 9 are proper factors of 18
- **Prime number**
  - Any whole number greater than zero that has exactly two factors – itself and one 2, 3, 5, 7, 11, 13, 17, 19…
- **Composite number**
  - Any whole number that has more than two factors
  - 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20…

Factor tree

- **Prime factor**
  - A factor that is also a prime number
- **Factor tree**
  - A tree that shows the prime factors of a number

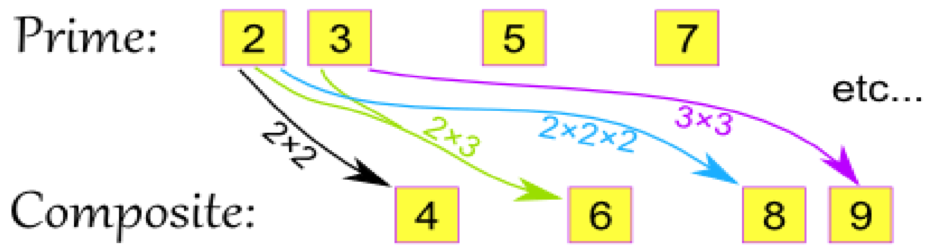




Prime Numbers

The **Basic Idea** is that any integer above 1 is either a Prime Number, or can be made by **multiplying Prime Numbers** together. Like this:

Click logo for link



This continues on:

- 10 is 2×5
- 11 is Prime,
- 12 is 2×2×3
- 13 is Prime
- 14 is 2×7
- 15 is 3×5
- 16 is 2×2×2×2
- 17 is Prime
- etc...

The fundamental theorem of arithmetic

http://www.mathsisfun.com/numbers/fundamental-theorem-arithmetic.html

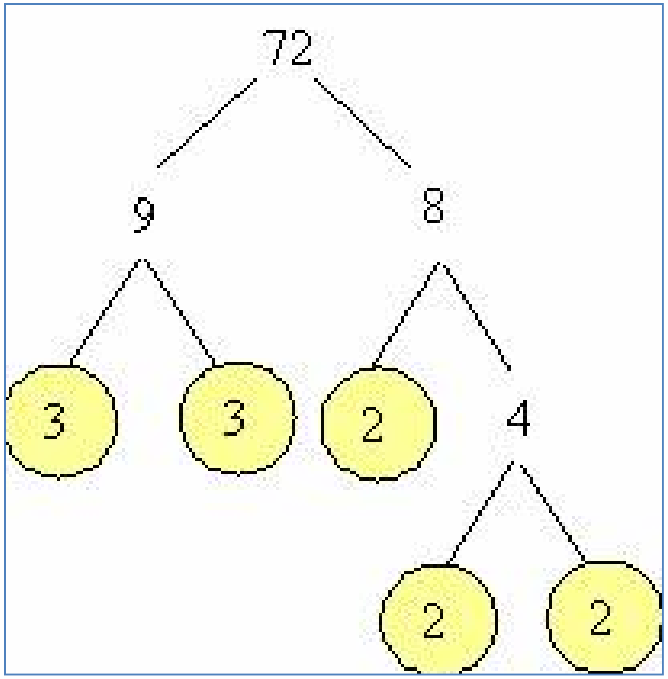
So they are either **prime**, or **primes multiplied together**

Your turn ….

1. Create a factor tree for the numbers 72
2. List all the factors for 120

Answers

1.



2. List all the factors for 120

- $120 = 1 \times 120$
- $120 = 2 \times 60$
- $120 = 3 \times 40$
- $120 = 4 \times 30$
- $120 = 5 \times 24$
- $120 = 6 \times 20$
- $120 = 8 \times 15$
- $120 = 10 \times 12$

∴ **Factors of 120**

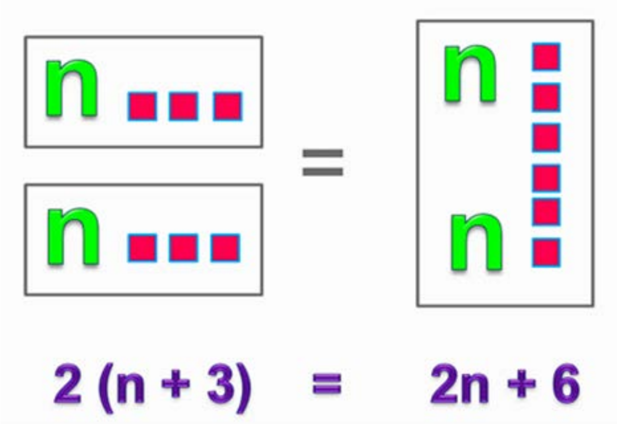
1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120

Expand

- Expanding and factorising are often used in algebra
- We '**distribute**' multiplication through addition or subtraction.
- Often referred to as either expanding the brackets or removing the brackets. For example:

Expand

- $2(n+3) \rightarrow 2 \times n + 2 \times 3$
- $\rightarrow 2n + 6$
- $\rightarrow 2(n+3) = 2n + 6$



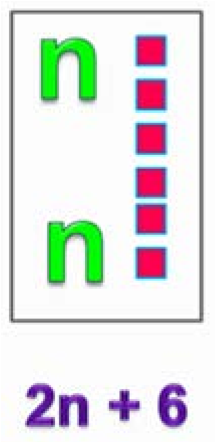
http://passyworldofmathematics.com/expanding-brackets-using-distributive-rule/

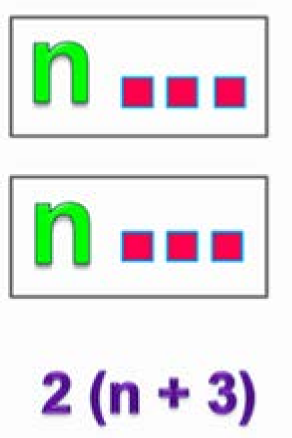
Factorise

- Involves working in the opposite direction (including brackets)

Factorise

- $2n + 6$ we know that $2n$ is a term with 2 factors, 2 and n the factors of 6 are – 1,2,3,6,
- The common factor for both terms is 2
  - 2 can be multiplied by n and 3 → $2 \times n = 2n + 2 \times 3$=6
- So we can take 2 outside brackets $2(n + 3)$





Your turn ….

Expand

1. $9(x + 2) =$
2. $2(a + 6 + c) =$

Factorise

1. $9 + 27c - 3a =$
2. $12 + 4a + 16 =$

Answers

Expand

1. $9(x + 2) = 9x + 18$
2. $2(a + 6 + c) = 2a + 12 + 2c$

Factorise

1. $9 + 27c - 3a = 3(3 + 9c - a)$
2. $12 + 4a + 16 = 4(3 + a + 4)$

Factorise

- Factorisation requires finding the highest common factor (HCF)
- For example: $5x + 15x^2 - 30x^3$ has three terms
  $5x \;\; and \;\; 15x^2 \;\; and \; 30x$

  All 3 terms have the same variable (x) and are a multiple of 5

  If $5x$ is a common factor, then

  $(5 \times x) + (5 \times 3 \times x \times x) - (5 \times 6 \times x \times x \times x)$

  $(5 \times x) + (5 \times 3 \times x \times x) - (5 \times 6 \times x \times x \times x)$
- Therefore, if we divide each term by $5x$ the HCF we end up with:

$$5x(1 + 3x - 6x^2)$$

Remember that if we divide a number by itself it equals one.

Let’s check

Is $5x + 15x^2 - 30x^3$ **the same as** $5x(1 + 3x - 6x^2)$?

Substitute $x \; for \; 2$

- $5 \times 2 + 15 \times 4 - 30 \times 8$
- $10 + 60 - 240$
- $70 - 240$
- -170

Substitute $x \; for \; 2$

- $10(1 + 6 - 6 \times 4)$
- $10(7 - 24)$
- 10×17
- -170

Factorising

Example Problem:

- Remove the Brackets:

$$5x(2+y) expands\ to\ (5x \times 2) + (5x \times y)$$
$$which\ is\ simplified\ to\ 10x + 5xy$$

- We can multiply by 2: remember the **commutative law**

$$-x(2x+6)\ expands\ to\ (-x \times 2x) + (-x \times 6)$$
$$which\ is\ simplified\ to - 2x^2 + (-6x)$$
$$\therefore -2x^2 - 6x$$

(Positive and negative make a negative)

Factorising

Factorise:

$$3x + 9x - x^2$$

The only thing in common is the variable x

If x^2 had a factor multiple of 3 as a coefficient, then we could factorise further

So we can take the variable outside of the brackets

$$x(3 + 9 - x)$$

Notice that we still have one x from x^2 inside the brackets

$\therefore x(12 - x)$

Test it $3x + 9x - x^2$ let's make $x = 2$

So $6 + 18 - 4 = 20$

Or $x(12 - x)$

$Let's\ make\ x = 2;\ 2 \times 10 = 20$

Factorising

Factorise $\mathbf{12x^3 + 4x^2 - 20x^4}$

What are the factors of all terms?

$$12xxx + 4xx - 20xxxx$$

We can see now that 4 is a **common factor** as is *x*

the HCFs: 4 and $x \times x$

$\therefore$ the **highest common factor** is $4x^2$

So we can factorise to get $\mathbf{4x^2(3x + 1 - 5x^2)}$

Your turn to check:

Is $12x^3 + 4x^2 - 20x^4$ **the same as** $4x^2(3x + 1 - 5x^2)$?

Your turn ….

Is $12x^3 + 4x^2 - 20x^4$ the same as $4x^2(3x + 1 - 5x^2)$?

Answers

Is $12x^3 + 4x^2 - 20x^4$ **the same as** $4x^2(3x + 1 - 5x^2)$ **?**

Let's substitute $x\ for\ 2$

$(12 \times 8) + (4 \times 4) - (20 \times 16)=$

$96 + 16 - 320 =$

$112 - 320 =$

-208

Let's substitute $x\ for\ 2$

$16\big(6 + 1 - (5 \times 4)\big) =$

$16(7 - 20) =$

$16 \times -13 =$

-208

Your turn …

A). Factorise by grouping

Example $2a+8 = 2 \times a + 2 \times 4 = 2(a+4)$

a) $6t + 3$

b) $8a + 20b$

c) $7k - 49$

B). Factorise fully

Example $x^2 - 7x = x \times x - 7 \times x = x(x-7)$

a) $t^2 - 5t$

b) $xy + 4y$

c) $p^2 + pq =$

d) $ab + ac + ad$

Answers

A). Factorise by grouping

a) $6t + 3 = 3(t + 1)$

b) $8a + 20b = 4(2a + 5b)$

c) $7k - 49 = 7(k - 7)$

B). Factorise fully

a) $t^2 - 5t = t(t - 5)$

b) $xy + 4y = y(x + 4)$

c) $p^2 + pq = p(p + q)$

d) $ab + ac + ad = a(b + c + d)$

Common Factors

- A common factor might be a combination of terms, such as a number, a term or several terms.
- For example, the term $4x$ is one term consisting of two factors
- And $3abc^2$ is also a term consisting of several factors

$3 \times a \times b \times c \times c$

So if we had $3abc^2 + 9a^2bc^3$ we could see that 3 is a common factor as is abc^2

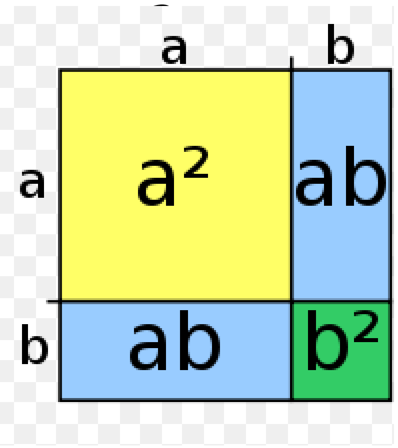
So we could factorise to $3abc^2(1 + 3a$c)

Identity: perfect squares

Always true for any numerical value

- From the square we can see that

  $(a+b)^2$ is the same as $a^2 + 2ab$



And

- $a(a+b) + b(b+a)$
- $(a+b)(a+b)$
- $a^2 + \mathrm{ab} + \mathrm{ba} + b^2$

- Let's use numerical values to check

Using the FOIL method

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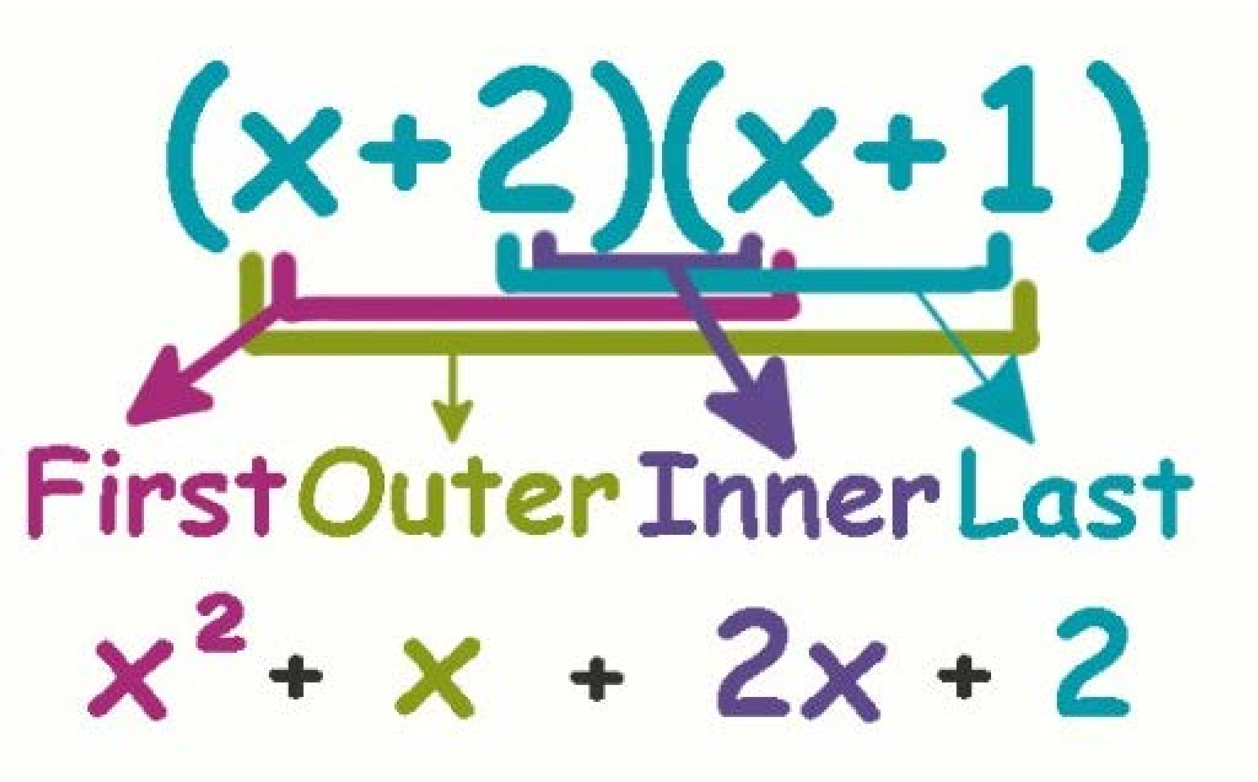
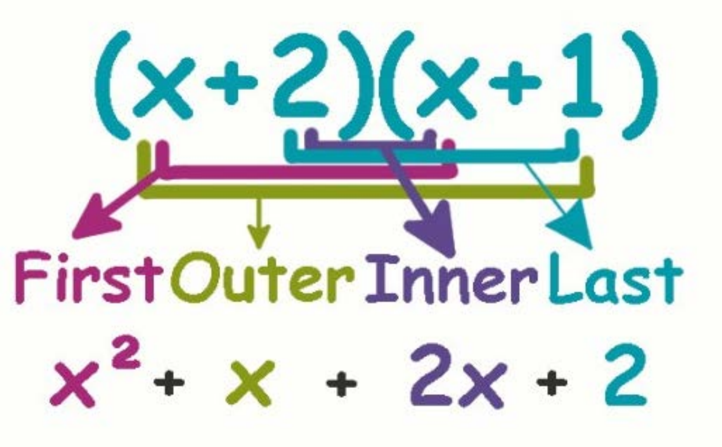


Your turn: expand & FOIL



- $20^2 = (10 + 10)^2$
- $20^2 = (15 + 5)^2$
- $20^2 = (18 + 2)^2 = (18 + 2)(18 + 2)$

Answers

- $20^2 = (10+10)^2$

  $(10+10)(10+10) = 100+100+100+100 = 400$

- $20^2 = (15+5)^2$

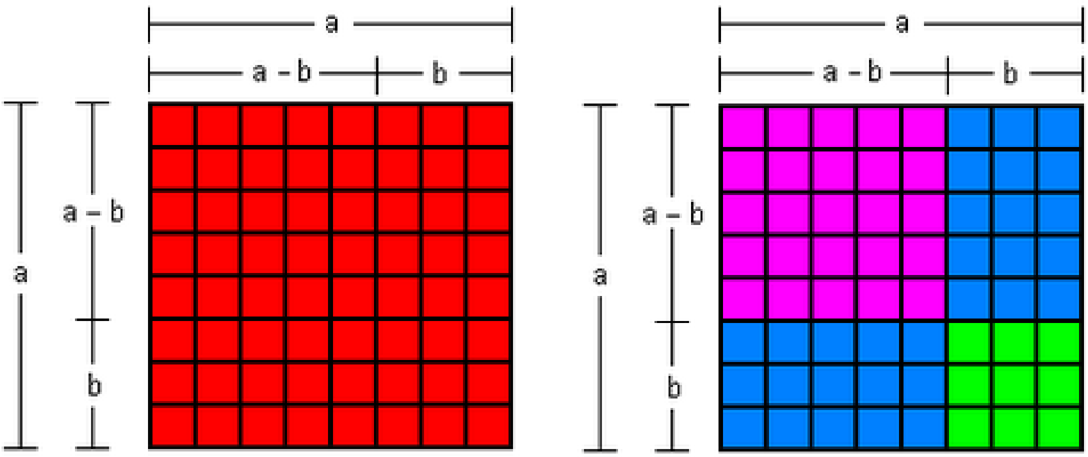
  $(15+5)(15+5) = 225+75+75+25 = 400$

- $20^2 = (18+2)^2 = (18+2)(18+2)$

  $324 + 36 + 36 + 4 = 400$

Difference of two squares

$$(a-b)^2 = a^2 - 2ab + b^2$$



$(a - b)^2$

$a^2 - ab - ab + b^2$

$a^2 - 2ab + b^2$

a = 8

b = 3

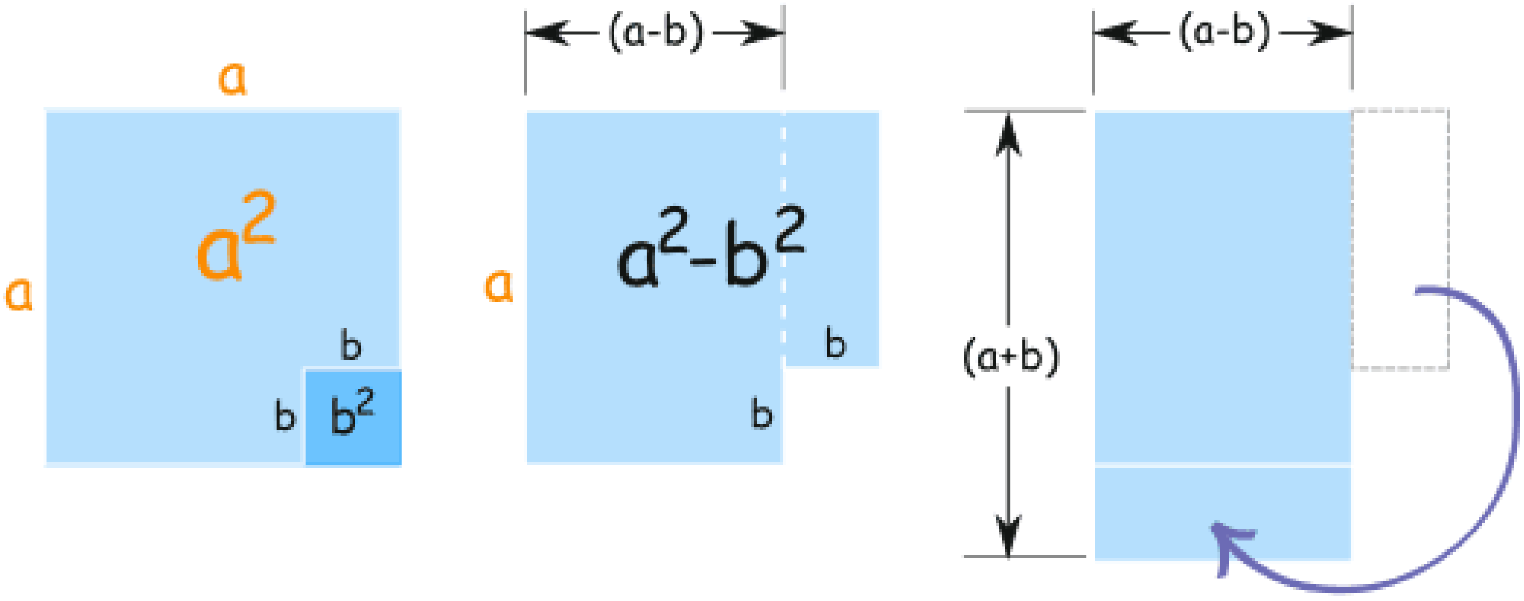
$(8 - 3)^2 = 25$

$8^2 = 64$

64 - 15 - 15 - 9 = 25

Difference of two squares

- $(a+b)(a-b) = a^2 - b^2$



Difference of two squares

Examples:

Factorise:

- $x^2 - 64 = (x - 8)(x + 8)$

- $9 - y^2 = (3 + y)(3 - y)$

Difference of two squares

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- A square might be the product of two or more terms
- For example:

$$16x^2 - 49y^2$$

- Let's factorise

$$(4x)^2 - (7\text{y})^2$$

$$=(4x + 7y)(4x - 7y)$$

Your turn …

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a) $3a^2 - 27$

b) $2x^2 - 72$

c) $5d^2 - 20c^2$

d) $16h^2 - 4$

e) $5t^2 - 180$

f) $7c^2 - 7d^2$

Answers

a) $3a^2 - 27 =$

$3(a^2 - 9)$=

$3(a - 3)(a + 3)$

b) $2x^2 - 72 =$

$2(x^2 - 36) =$

$2(x - 6)(x + 6)$

c) $5d^2 - 20c^2 =$

$5(d^2 - 4c^2)$=

$5(d - 2c)(d + 2c)$

d) $16h^2 - 4 =$

$4(4h^2 - 1) =$

$4(2h - 1)(2h + 1)$

e) $5t^2 - 180 =$

$5(t^2 - 36) =$

$5(t - 6)(t + 6)$

f) $7c^2 - 7d^2$

$7(c^2 - d^2)$

$7(c + d)(c - d)$

Divisibility rules

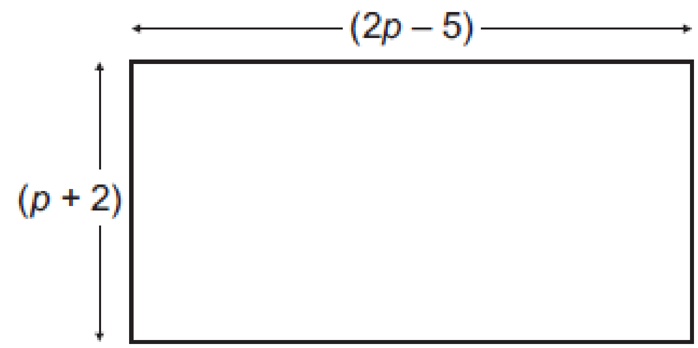
- Wouldn't it just be easier to use the 7 divisibility trick to determine if it's a multiple of 14?
- All you do is double the last digit of 154, (you double the 4 to get 8) subtract that 8 from the remaining truncated number (15), giving the result of 7.
- 7 is obviously a multiple of 7, meaning the original number is divisible by 7.
- Since the number 154 is also even, meaning 2's a factor it's divisible by 14 (which has 7 and 2 as factors).
- That sounds easier than dividing. There are a bunch more divisibility tricks for all prime numbers up to 50 here: http://www.savory.de/maths1.htm

Revision 1

- The perimeter of a rectangle is 90cm
- What is the value of *p*

The perimeter of this rectangle is 90 cm.
What is the value of *p*?

14 ⬭ 16 ⬭ 29 ⬭ 31 ⬭



QSA (2011)

Revision 2

- Which two expressions are equivalent?

Which one of the following expressions is equivalent to $3(2n + 4)$?

$6n + 4$ ⬭ $5n + 12$ ⬭ $2(3n + 6)$ ⬭ $5n + 7$ ⬭

QSA (2011)

Revision 3

- What is the value of *k*?

$$\frac{3k}{5} = k + 6$$

What is the value of **k** in this equation?

k = ☐

QSA (2011)

Revision 4

- Based on the information in the table, what is the value of y when $x = -2$

This table shows the values of y for some different values of x when $y = 4 + x - 2x^2$.

x	– 3	– 2	– 1	0	1	2
y	– 17	?	1	4	3	– 2

What is the value of y when $x = -2$?

QSA (2011)

Revision 5

- Substitute and solve:

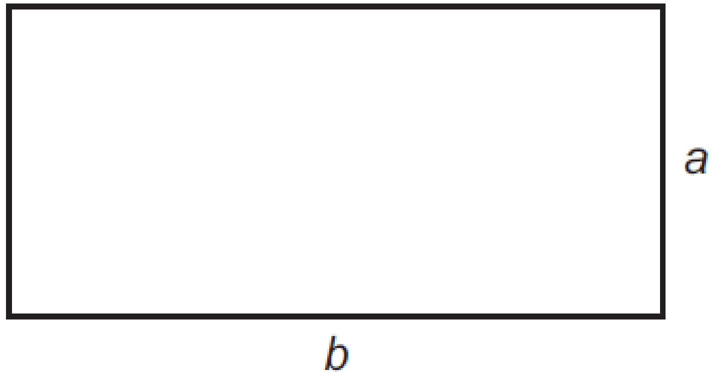
When $k = 3$ and $p = -1$ what is the value of $\dfrac{2k^2 + 3}{5 - 2p}$?

3	5	7	13

QSA (2011)

Revision 6

A square corner measuring x cm by x cm is cut out of this rectangular sheet of paper.



Which of these statements is true for the piece of paper that remains?

- Its area is $ab - x^2$ and its perimeter is $2a + 2b$.
- Its area is $(ab - x)^2$ and its perimeter is $2(a+b)$.
- Its area is $ab - x^2$ and its perimeter is $2(a + b - x)$.
- Its area is $(ab - x)^2$ and its perimeter is $2a + 2b - 4x$.

Revision 1 answer

perimeter = 2(length + width).

$\therefore \quad 2(p+2) + 2(2p-5) = 90\text{cm}.$

$2p+4 + 4p-10 = 90\text{cm}$

$6p+4 - 10 = 90\text{cm}$

$6p+4(-10+10) = 90+10$

$6p+4 = 100$

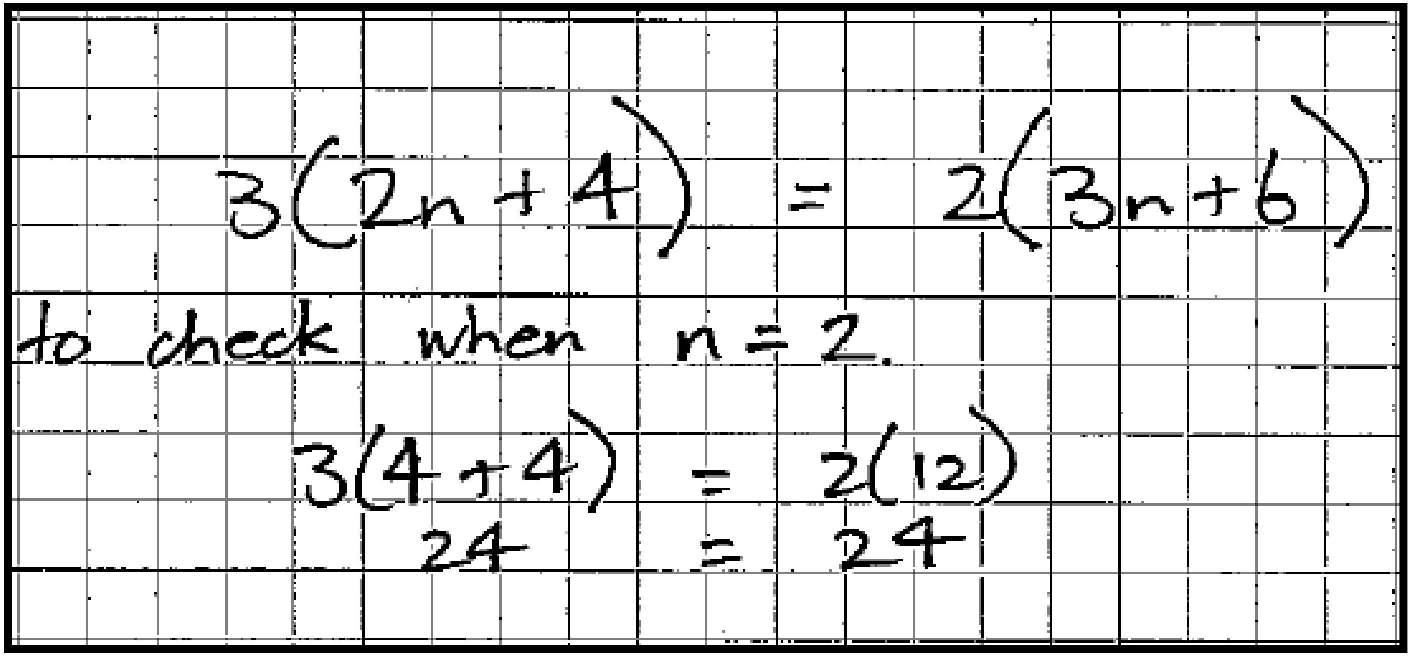
$6p = 100 - 4$

$6p = 96$

$\therefore \quad p = 96 \div 6$

$p = 16$

Revision 2 answer



Revision 3 answer

$$\frac{3k}{5} = k + 6$$

$$3k = 5k + 30 \quad \text{(multiply by 5)}$$

$$3k - 30 = 5k$$

$$-30 = 5k - 3k$$

$$-30 = 2k$$

$$\frac{-30}{2} = k$$

$$\therefore \quad k = -15$$

Revision 4 answer

$$y = 4 + x - 2x^2 \quad (x = -2)$$

$$y = 4 + (-2) - 2(-2^2)$$

$$y = 4 - 2 - 2(4)$$

$$y = 4 - 2 - 8$$

$$y = 2 - 8$$

$$y = -6$$

Revision 5 answer

$$\frac{2k^2+3}{5-2p} \qquad k=3,\ p=-1$$

$$= \frac{2(3^2)+3}{5-(2\times -1)}$$

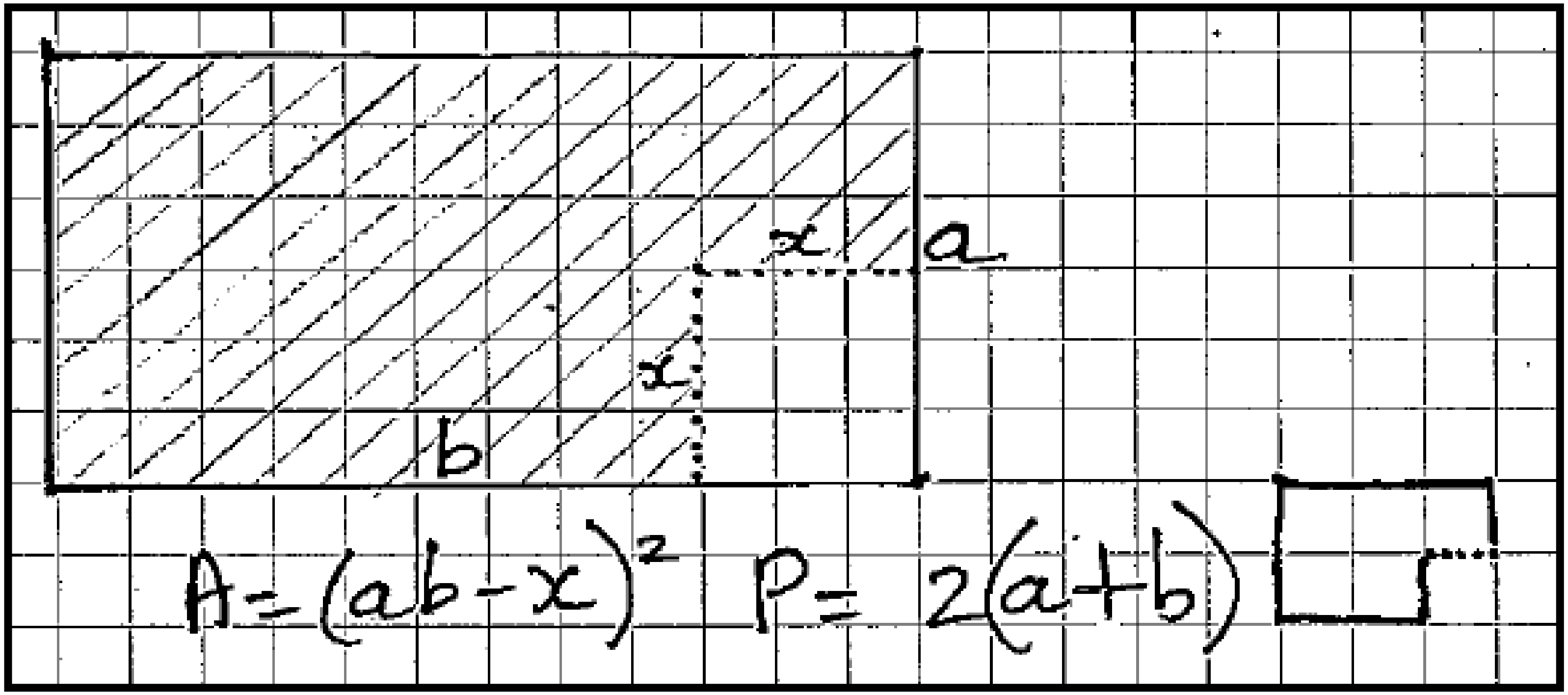
$$= \frac{2(9)+3}{5-(-2)}$$

$$= \frac{18+3}{5+2}$$

$$= \frac{21}{7}$$

$$\therefore = 3$$

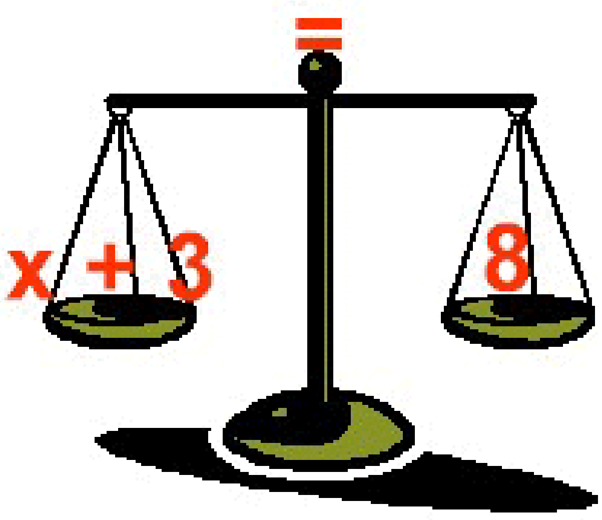
Revision 6 answer



Expanding and Factorising

Reflect on the learning intentions ….

- Recap
- Expanding equations
- Factorising equations
- Identity: perfect pairs
- Difference of two squares



Resources

- https://www.khanacademy.org/math/arithmetic/factors-multiples/divisibility_and_factors/v/finding-factors-and-multiples?utm_medium=email&utm_content=5&utm_campaign=khanacademy&utm_source=digest_html&utm_term=thumbnail
- Baker, L. (2000). *Step by step algebra 1 workbook.* NSW: Pascal Press
- Baker, L. (2000). *Step by step algebra 2 workbook.* NSW: Pascal Press
- Queensland Studies Authority. (2011). *2011 NAPLAN: Year 9 numeracy. Brisbane:* Queensland Government